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FINITE PLANE STRAIN

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CONTENTS

A general theory of plane strain, valid for large elastic deformations of isotropic materials, is developed using a general system of co-ordinates. No restriction is imposed upon the form of the strain-energy function in the formulation of the basic theory, apart from that arising naturally from the assumption of plane strain. In applications, attention is confined to incompressible materials, and the general method of approach is illustrated by the examination of a number of problems which are capable of exact solution. These include the flexure of a cuboid, and of an initially curved cuboid, and a generalization of the shear problem.

A method of successive approximation is then evolved, suitable for application to problems for which exact solutions are not readily obtainable. Attention is again confined to incompressible materials, and the approximation process is terminated when the second-order terms have been obtained. In considering problems in plane strain, complex variable techniques are employed and the stress and displacement functions are expressed in terms of complex potential functions. In dealing with finite elastic deformations, a complex co-ordinate system may be chosen which is related either to points in the deformed body or to points in the undeformed body, and in the present paper both methods are developed. The theory is applied to obtain solutions for an infinite body which contains either a circular hole or a circular rigid inclusion, and which is under a uniform tension at infinity.

1. Introduction

In the theoretical treatment of large elastic deformations of highly elastic, incompressible, isotropic materials, a number of problems have been solved completely by Rivlin (1948*a* to 1953) and by Green & Shield (1950, 1951), without any restriction either upon the form of the strain-energy function for the elastic material, or upon the magnitude of the deformation. The determination of these exact solutions has depended essentially upon the ability to choose suitable co-ordinate systems in which to specify the initial and final configurations of the elastic body, so that in the subsequent analysis, partial differential equations can be avoided. Further solutions for thin sheets of isotropic materials have been obtained by numerical methods by Rivlin & Thomas (1951) and by Adkins & Rivlin (1952), but here again the successful treatment of such problems has depended upon the existence of conditions of symmetry which render the unknown quantities functions of one independent variable only.

In the absence of such symmetry conditions, or when a simplifying choice of co-ordinate systems does not appear possible, it is natural to consider methods of successive approximation, by means of which solutions can be obtained, which, although not valid for an unrestricted range of deformation, nevertheless provide an extension of the results of the classical infinitesimal theory of elasticity. Second approximation solutions for comparatively simple deformations of compressible materials have been given by Murnaghan (1937, 1951), and by Green & Wilkes (1953), and second-order effects in the torsion of cylinders of incompressible materials have been considered by Green & Shield (1951) using complex variable techniques. More recently Rivlin (1953), using a rectangular Cartesian co-ordinate system, has developed a general method for the solution of problems in second-order elasticity theory, and has applied his method to consider the torsion of cylindrical tubes and rods of compressible and incompressible materials.

In the present paper a general method of successive approximation is evolved for problems in plane strain for which there are various simplifying features. In particular, it is shown that the assumption of plane strain implies a relationship between the strain invariants, as noticed by Murnaghan (1951), and for an incompressible material the strain-energy function may then be considered as a function of a single strain invariant I. The introduction of the Airy stress function ϕ to satisfy the equations of equilibrium results in a further simplification of the theory. The work of §§ 4 and 5 on the Airy stress function, valid for finite deformations, follows the development given by Green & Zerna† for classical theory.

Before proceeding to approximation methods the general theory of plane strain superposed on uniform extension is developed in a general co-ordinate system for large elastic deformations (§§ 3 to 5). In subsequent sections attention is confined to incompressible materials, and to illustrate the general method of approach several problems are considered which are capable of exact solution. These include the problem of simple flexure (§ 6) previously treated by Rivlin (1949 *b*, *c*) and a generalization of this problem (§ 7) which contains, as special cases, the inflation and the eversion of a cylindrical tube. The latter problems have also been solved in a different manner by Rivlin (1949 *c*). In the following section (§ 8) the generalized shear of a cuboid of incompressible material is considered.

In developing a method of successive approximation, it is assumed that the stress and displacement functions may be expressed as power series in a characteristic real parameter ϵ, the choice of this parameter depending upon the problem under consideration. By expanding the equations of equilibrium, incompressibility condition and boundary conditions in terms of this parameter, and considering separately the coefficients of corre-

† *Theoretical elasticity* (in the Press).

sponding powers of ϵ in these equations, a series of relations is obtained for the determination of successive terms in the expansions for the stress and displacement functions. In the present paper attention is confined to terms of the first and second orders only, for which the form of strain-energy function suggested by Mooney (1940) is sufficiently general, but in principle the process may be continued until any desired degree of approximation is obtained.

For two-dimensional problems there are considerable advantages in formulating the theory in complex variable notation. Explicit expressions for the stress and displacement functions can then be obtained in terms of complex potential functions, two additional functions being introduced for each succeeding stage of the approximation process. These functions are chosen to satisfy the boundary conditions for the problem under consideration, and this procedure forms a natural extension of the corresponding methods of the classical infinitesimal theory of elasticity (see, for example, Muschelisvili 1932, 1933; Green 1942; Stevenson 1943, 1945, etc.). In considering finite deformations, however, the complex co-ordinate system may be related either to points in the undeformed body or to points in the deformed body, the choice for any particular problem depending upon the nature of the boundary conditions. The latter choice of co-ordinate system yields equations which are somewhat simpler in form and is considered in §§ 9 and 10. The corresponding theory for complex co-ordinates in the undeformed body is developed in §§ 11 and 12.

In the final sections of the paper the theory is applied to obtain solutions for an infinite body which contains either a circular hole or a circular rigid inclusion, and which is subjected to a uniform tension at infinity.

2. Notation and formulae

We briefly summarize the notation and formulae which have been used by Green & Zerna (1950) and Green & Shield (1950, 1951) and which will be required in the present paper.

The points of an unstrained and unstressed body at rest at time $t=0$ are defined by a system of rectangular Cartesian co-ordinates x_i or by a general curvilinear system of co-ordinates θ_i. The curvilinear co-ordinates θ_i move with the body as it is deformed and form a curvilinear system in the strained body at time t. The covariant and contravariant metric tensors for the co-ordinate system θ_i in the unstrained body are denoted by g_{ij} and g^{ij} respectively, and for the co-ordinate system in the strained body, at time t, the corresponding metric tensors are G_{ij} and G^{ij} respectively, whilst

$$g = |g_{ij}|, \quad G = |G_{ij}|. \tag{2·1}$$

Latin indices take the values 1, 2, 3. The strain invariants are

$$I_1 = g^{ij}G_{ij}, \quad I_2 = I_3 g_{ij} G^{ij}, \quad I_3 = G/g, \tag{2·2}$$

and the contravariant stress tensor (per unit area of the strained body) referred to θ_i co-ordinates in the strained body can be expressed in the form

$$\tau^{ij} = g^{ij}\Phi + B^{ij}\Psi + G^{ij}p, \tag{2·3}$$

for a body which is isotropic and homogeneous in its undeformed state, where

$$\Phi = \frac{2}{\sqrt{I_3}}\frac{\partial W}{\partial I_1}, \quad \Psi = \frac{2}{\sqrt{I_3}}\frac{\partial W}{\partial I_2}, \quad p = 2\sqrt{I_3}\frac{\partial W}{\partial I_3}, \tag{2·4}$$

$$B^{ij} = g^{ij}I_1 - g^{ir}g^{js}G_{rs} = \frac{1}{g}\,\mathrm{e}^{irm}\,\mathrm{e}^{jsn}\,g_{rs}\,G_{mn}, \tag{2·5}$$

and

$$W = W(I_1, I_2, I_3), \tag{2·6}$$

is the strain-energy function measured per unit volume of the unstrained body. Also e^{irm} is equal to $+1$ or -1 according as i, r, m is an even or odd permutation of 1, 2, 3, and equal to 0 otherwise.

With the above notation one form of the equations of equilibrium is

$$\tau^{ij}\|_i = 0 \tag{2·7}$$

when body forces are zero, where the double line denotes covariant differentiation with respect to the deformed body, that is, with respect to θ_i and the metric tensor components G_{ij}, G^{ij}. For this covariant differentiation we need the Christoffel symbols

$$\Gamma^r_{ij} = \tfrac{1}{2}G^{rs}(G_{si,j} + G_{sj,i} - G_{ij,s}), \tag{2·8}$$

where a comma denotes partial differentiation with respect to θ_i. An alternative form of the equations of equilibrium which is needed in this paper is

$$\mathbf{T}_{i,i} = 0, \tag{2·9}$$

where

$$\mathbf{T}_i = \sqrt{(GG^{ii})}\,\mathbf{t}_i = \sqrt{(G)}\,\tau^{ij}\mathbf{E}_j, \tag{2·10}$$

and $\mathbf{E}_j$ and $\mathbf{E}^j$ are covariant and contravariant base vectors in the deformed body. In (2·10) $\mathbf{t}_i$ denotes the stress vector associated with the surface $\theta_i = $ constant. If $\mathbf{t}$ is the stress vector associated with a surface in the deformed body whose unit normal $\mathbf{n}$ is given by

$$\mathbf{n} = n_i\mathbf{E}^i, \tag{2·11}$$

then

$$\mathbf{t} = \frac{n_i\mathbf{T}_i}{\sqrt{G}} = \sum_i n_i\mathbf{t}_i\sqrt{G^{ii}}. \tag{2·12}$$

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3. Plane strain superposed on uniform extension

The points of the strained body may also be defined by a set of rectangular Cartesian co-ordinates y_i, and we take the y_3-axis to be parallel to the x_3-axis which is in the undeformed body, so that the plane $y_3 = 0$ corresponds to the plane $x_3 = 0$. We now suppose that the body is deformed by a uniform finite extension parallel to the x_3-axis, with constant extension ratio λ, and that subsequently the body receives a finite plane strain parallel to the (x_1, x_2) plane. Thus if we choose the moving curvilinear co-ordinate θ_3 so that $\theta_3 = y_3$ then

$$x_3 = y_3/\lambda = \theta_3/\lambda, \tag{3·1}$$

and

$$x_\alpha = x_\alpha(\theta_1, \theta_2), \quad y_\alpha = y_\alpha(\theta_1, \theta_2, t), \tag{3·2}$$

greek indices taking the values 1, 2. It follows from (3·1) and (3·2) that

$$g_{ij} = \begin{pmatrix} a_{11}, & a_{12}, & 0 \\ a_{12}, & a_{22}, & 0 \\ 0, & 0, & \frac{1}{\lambda^2} \end{pmatrix}, \quad g^{ij} = \begin{pmatrix} a^{11}, & a^{12}, & 0 \\ a^{12}, & a^{22}, & 0 \\ 0, & 0, & \lambda^2 \end{pmatrix}, \quad g = \frac{a}{\lambda^2}, \tag{3·3}$$

$$G_{ij} = \begin{pmatrix} A_{11}, & A_{12}, & 0 \\ A_{12}, & A_{22}, & 0 \\ 0, & 0, & 1 \end{pmatrix}, \quad G^{ij} = \begin{pmatrix} A^{11}, & A^{12}, & 0 \\ A^{12}, & A^{22}, & 0 \\ 0, & 0, & 1 \end{pmatrix}, \quad G = A, \tag{3·4}$$

where $a_{\alpha\beta}$, $a^{\alpha\beta}$ are the covariant and contravariant metric tensors associated with curvilinear co-ordinates θ_α in a plane $x_3 = 0$ of the undeformed body; $A_{\alpha\beta}$, $A^{\alpha\beta}$ are the covariant and contravariant metric tensors associated with curvilinear co-ordinates θ_α in a plane $y_3 = 0$ of the deformed body, and

$$a = |a_{\alpha\beta}|, \quad A = |A_{\alpha\beta}|. \tag{3·5}$$

It follows from (2·2), (3·3) and (3·4) that the strain invariants are given by

$$\left.\begin{aligned} I_1 &= \lambda^2 + a^{\alpha\beta} A_{\alpha\beta}, \\ I_2 &= \lambda^2 (A/a)\, a_{\alpha\beta} A^{\alpha\beta} + A/a, \\ I_3 &= \lambda^2 A/a. \end{aligned}\right\} \tag{3·6}$$

These invariants are not, however, independent, for

$$\begin{aligned} a_{\alpha\beta} A^{\alpha\beta} A/a &= ({}_0\epsilon_{\alpha\rho})\,({}_0\epsilon_{\beta\nu})\,(\epsilon^{\alpha\lambda}\epsilon^{\beta\mu})\, a^{\rho\nu} A_{\lambda\mu} A/a \\ &= \epsilon_{\alpha\rho}\epsilon_{\beta\nu}\epsilon^{\alpha\lambda}\epsilon^{\beta\mu} a^{\rho\nu} A_{\lambda\mu} \\ &= a^{\alpha\beta} A_{\alpha\beta}, \end{aligned}$$

where

$$\left.\begin{aligned} {}_0\epsilon^{\alpha\rho}\sqrt{a} = \frac{{}_0\epsilon_{\alpha\rho}}{\sqrt{a}} = \epsilon^{\alpha\rho}\sqrt{A} = \frac{\epsilon_{\alpha\rho}}{\sqrt{A}} &= \quad 1 \quad \text{if} \quad \alpha = 1,\ \rho = 2, \\ &= -1 \quad \text{if} \quad \alpha = 2,\ \rho = 1, \\ &= 0 \text{ otherwise}, \end{aligned}\right\} \tag{3·7}$$

and

$$\epsilon_{\alpha\rho}\epsilon^{\alpha\lambda} = \delta^\lambda_\rho, \tag{3·8}$$

δ^λ_ρ being, as usual, the Kronecker delta. Thus

$$I_2 = \lambda^2 (I_1 - \lambda^2) + I_3/\lambda^2. \tag{3·9}$$

The tensor B^{ij} may be calculated from (2·5), (3·3), (3·4) and (3·6), and is

$$\left.\begin{aligned} B^{\alpha\beta} &= ({}_0\epsilon^{\alpha\rho})\,({}_0\epsilon^{\beta\nu})\, a_{\rho\nu}\lambda^2 + \epsilon^{\alpha\rho}\epsilon^{\beta\nu} A_{\rho\nu} A/a \\ &= \lambda^2 a^{\alpha\beta} + A A^{\alpha\beta}/a, \\ B^{\alpha 3} &= 0, \\ B^{33} &= \lambda^2 (I_1 - \lambda^2). \end{aligned}\right\} \tag{3·10}$$

It follows from (2·3), (2·4), (3·3), (3·4) and (3·10) that the stress tensor $\tau^{\alpha\beta}$ becomes

$$\begin{aligned} \tau^{\alpha\beta} &= a^{\alpha\beta}\Phi + B^{\alpha\beta}\Psi + A^{\alpha\beta} p \\ &= \frac{2}{\sqrt{I_3}}\left(\frac{\partial W}{\partial I_1} + \lambda^2 \frac{\partial W}{\partial I_2}\right) a^{\alpha\beta} + 2\sqrt{(I_3)}\left(\frac{\partial W}{\partial I_3} + \frac{1}{\lambda^2}\frac{\partial W}{\partial I_2}\right) A^{\alpha\beta}. \end{aligned} \tag{3·11}$$

If we put
$$\left.\begin{aligned} I_1 &= I = \lambda^2 + a^{\alpha\beta}A_{\alpha\beta}, \\ I_3 &= J = \lambda^2 A/a, \end{aligned}\right\} \tag{3·12}$$

then, from (3·9),
$$I_2 = \lambda^2(I-\lambda^2) + J/\lambda^2, \tag{3·13}$$

and the strain-energy function (2·6) reduces to a function of two strain invariants I, J so that†
$$W = W(I, J). \tag{3·14}$$

Hence
$$\frac{\partial W}{\partial I_1} + \lambda^2 \frac{\partial W}{\partial I_2} = \frac{\partial W}{\partial I} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial I} = \frac{\partial W}{\partial I},$$
$$\frac{\partial W}{\partial I_3} + \frac{1}{\lambda^2}\frac{\partial W}{\partial I_2} = \frac{\partial W}{\partial J} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial J} = \frac{\partial W}{\partial J},$$

and so (3·11) may be written
$$\tau^{\alpha\beta} = a^{\alpha\beta}H + A^{\alpha\beta}K, \tag{3·15}$$

where
$$H = \frac{2}{\sqrt{J}}\frac{\partial W}{\partial I}, \quad K = 2\sqrt{J}\frac{\partial W}{\partial J}, \tag{3·16}$$

and W is given by (3·14). The functions H and K are invariants depending only on I and J, and we see that
$$\Phi + \lambda^2\Psi = H, \quad p + I_3\Psi/\lambda^2 = K, \tag{3·17}$$

where Φ, Ψ, p are given by (2·4) in terms of the strain-energy function (2·6) which is regarded as a function of the three invariants I_1, I_2, I_3. On the other hand, H, K are given by (3·16) in terms of the strain-energy function (3·14) which is regarded as a function of the two invariants I, J.

From (2·3), (2·4), (3·3), (3·4) and (3·10) we see that the remaining stress components $\tau^{\alpha 3}$ are zero and that
$$\begin{aligned} \tau^{33} &= \lambda^2\Phi + \lambda^2(I_1-\lambda^2)\,\Psi + p \\ &= (2\lambda^2 - I + J/\lambda^4)\,\Phi + (I-\lambda^2-J/\lambda^4)\,H + K, \end{aligned} \tag{3·18}$$

if we also use (3·17).

If the body is incompressible then
$$J = 1 \quad \text{or} \quad \lambda^2 A = a, \tag{3·19}$$

and W becomes a function of I_1, I_2 or a function of a single invariant I. The stress-strain relation (3·15) is still valid, but in this case
$$H = 2\frac{\mathrm{d}W(I)}{\mathrm{d}I}, \tag{3·20}$$

and K is a scalar invariant function which has to be determined from the equations of equilibrium and the boundary conditions. Also, (3·18) becomes
$$\tau^{33} = (2\lambda^2 - I + 1/\lambda^4)\,\Phi + (I-\lambda^2-1/\lambda^4)\,H + K, \tag{3·21}$$

where
$$\Phi = 2\frac{\partial W(I_1, I_2)}{\partial I_1}. \tag{3·22}$$

If the strain energy $W(I_1, I_2)$ takes the approximate Mooney form
$$W(I_1, I_2) = C_1(I_1-3) + C_2(I_2-3), \tag{3·23}$$

† W also depends on λ, but this is assumed to be a known constant.

where C_1, C_2 are constants then, using (3·13), we have

$$W(I) = (C_1+C_2\lambda^2)(I-3)-C_2(\lambda^2-1)^3/\lambda^2, \tag{3·24}$$

and therefore
$$\Phi = 2C_1, \quad H = 2(C_1+C_2\lambda^2), \tag{3·25}$$

$$\tau^{33} = 2\lambda^2 C_1 + 2\lambda^2(I-\lambda^2-1/\lambda^4)\,C_2+K. \tag{3·26}$$

4. Airy's stress function

From (3·3) and (3·4) we see that the metric tensors of the unstrained and strained bodies are dependent only on the two co-ordinates θ_α. Also the base vectors $\mathbf{E}_\alpha$, $\mathbf{E}^\alpha$ are dependent only on θ_α and are parallel to the plane $y_3 = 0$. The base vectors $\mathbf{E}_3$, $\mathbf{E}^3$ are constant in direction (the y_3-axis) and are of unit magnitudes. The stress tensor τ^{ij} which is given by (3·15) and (3·18) is dependent only on θ_α and, from (2·10),

$$\left.\begin{aligned}\mathbf{T}_\alpha &= \sqrt{(AA^{\alpha\alpha})}\,\mathbf{t}_\alpha = \sqrt{(A)}\,\tau^{\alpha\beta}\mathbf{E}_\beta,\\ \mathbf{T}_3 &= \sqrt{(A)}\,\mathbf{t}_3 \qquad = \sqrt{(A)}\,\tau^{33}\mathbf{E}_3,\end{aligned}\right\} \tag{4·1}$$

so that $\mathbf{T}_i$ is dependent only on θ_α. The equation of equilibrium (2·9) therefore reduces to

$$\mathbf{T}_{\alpha,\alpha} = 0. \tag{4·2}$$

This equation can be satisfied by

$$\mathbf{T}_\alpha = \sqrt{(A)}\,\epsilon^{\gamma\alpha}\boldsymbol{\chi}_{,\gamma}, \tag{4·3}$$

where $\boldsymbol{\chi}$ is a vector in the plane $y_3 = 0$. If

$$\boldsymbol{\chi} = \chi^\beta\mathbf{E}_\beta, \tag{4·4}$$

then
$$\boldsymbol{\chi}_{,\gamma} = \chi^\beta\|_\gamma\,\mathbf{E}_\beta, \tag{4·5}$$

and hence
$$\mathbf{T}_\alpha = \sqrt{(A)}\,\epsilon^{\gamma\alpha}\chi^\beta\|_\gamma\,\mathbf{E}_\beta. \tag{4·6}$$

Covariant differentiation is here with respect to the plane $y_3 = 0$ in the deformed body using Christoffel symbols

$$\Gamma^\alpha_{\beta\gamma} = \tfrac{1}{2}A^{\alpha\rho}\{A_{\beta\rho,\gamma}+A_{\gamma\rho,\beta}-A_{\beta\gamma,\rho}\}. \tag{4·7}$$

We observe, in passing, that the order of covariant differentiations in this plane is still immaterial since the Riemann Christoffel tensor in the plane vanishes. From (4·1) and (4·6) it follows that

$$\tau^{\alpha\beta} = \epsilon^{\gamma\alpha}\chi^\beta\|_\gamma, \tag{4·8}$$

and since $\tau^{\alpha\beta}$ is symmetrical we may put

$$\chi^\beta = \epsilon^{\rho\beta}\phi_{,\rho}. \tag{4·9}$$

Hence
$$\tau^{\alpha\beta} = \epsilon^{\gamma\alpha}\epsilon^{\rho\beta}\phi\|_{\rho\gamma} = \epsilon^{\alpha\gamma}\epsilon^{\beta\rho}\phi\|_{\gamma\rho}, \tag{4·10}$$

where ϕ is a scalar invariant function of θ_1, θ_2 and is Airy's stress function for the deformed body.

Equation (4·10) may be solved for $\phi\|_{\alpha\beta}$ to give

$$\phi\|_{\alpha\beta} = \epsilon_{\alpha\gamma}\epsilon_{\beta\rho}\tau^{\gamma\rho} = (A/a)\,({}_0\epsilon_{\alpha\gamma})\,({}_0\epsilon_{\beta\rho})\,\tau^{\gamma\rho}. \tag{4·11}$$

Hence, from (3·15) and (4·11),

$$\begin{aligned}\phi\|_{\alpha\beta} &= (A/a)\,({}_0\epsilon_{\alpha\gamma})\,({}_0\epsilon_{\beta\rho})\,a^{\gamma\rho}H+\epsilon_{\alpha\gamma}\epsilon_{\beta\rho}A^{\gamma\rho}K\\ &= (A/a)\,a_{\alpha\beta}H+A_{\alpha\beta}K.\end{aligned} \tag{4·12}$$

For an incompressible body, using (3·19), we see that (4·12) reduces to

$$\phi\,\|_{\alpha\beta} = a_{\alpha\beta}H/\lambda^2 + A_{\alpha\beta}K. \tag{4·13}$$

We may eliminate K from these equations. Thus

$$2K + (I-\lambda^2)\,H = A^{\alpha\beta}\phi\,\|_{\alpha\beta} = \phi\,\|_{\alpha}^{\alpha} = \frac{1}{\surd A}\{\surd(A)\,A^{\rho\gamma}\phi,_{\rho}\},_{\gamma}, \tag{4·14}$$

so that
$$2\phi\,\|_{\alpha\beta} = A_{\alpha\beta}\phi\,\|_{\rho}^{\rho} + \{2a_{\alpha\beta}/\lambda^2 - (I-\lambda^2)\,A_{\alpha\beta}\}\,H, \tag{4·15}$$

or
$$2\phi\,\|_{\beta}^{\alpha} = \delta_{\beta}^{\alpha}\phi\,\|_{\rho}^{\rho} + \{2A^{\alpha\gamma}a_{\beta\gamma}/\lambda^2 - (I-\lambda^2)\,\delta_{\beta}^{\alpha}\}\,H. \tag{4·16}$$

Also from (3·21) and (4·14) we have

$$2\tau^{33} = 2(2\lambda^2 - I + 1/\lambda^4)\,\Phi + (I-\lambda^2-2/\lambda^4)\,H + \phi\,\|_{\alpha}^{\alpha}. \tag{4·17}$$

For a Mooney material with strain-energy in the approximate form (3·23), equations (4·15), (4·16) and (4·17) become, respectively,

$$2\phi\,\|_{\alpha\beta} = 2(C_1+\lambda^2C_2)\,\{2a_{\alpha\beta}/\lambda^2 - (I-\lambda^2)\,A_{\alpha\beta}\} + A_{\alpha\beta}\phi\,\|_{\rho}^{\rho}, \tag{4·18}$$

$$2\phi\,\|_{\beta}^{\alpha} = 2(C_1+\lambda^2C_2)\,\{2A^{\alpha\rho}a_{\rho\beta}/\lambda^2 - (I-\lambda^2)\,\delta_{\beta}^{\alpha}\} + \delta_{\beta}^{\alpha}\phi\,\|_{\rho}^{\rho}, \tag{4·19}$$

and
$$2\tau^{33} = 2(3\lambda^2 - I)\,C_1 + 2\lambda^2(I-\lambda^2-2/\lambda^4)\,C_2 + \phi\,\|_{\alpha}^{\alpha}. \tag{4·20}$$

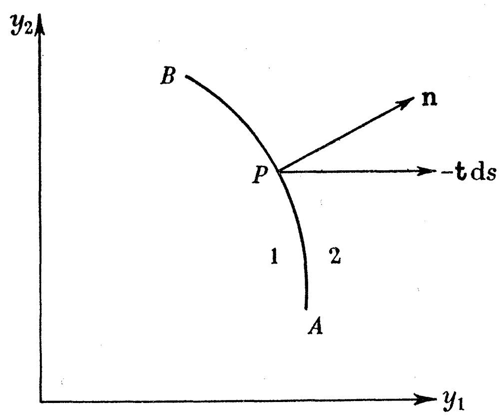

FIGURE 1.

5. FORCE AND COUPLE RESULTANTS

Consider any curve AB (figure 1) which lies in the plane $y_3 = 0$ of the deformed body and which does not intersect itself, and suppose that positive direction along the curve is from A to B. The curve separates the plane into two regions 1 and 2 which are immediately adjacent to AB, and the force exerted by region 1 on the region 2 across an element ds of AB is $-\mathbf{t}\,\mathrm{d}s$ measured per unit length of the y_3-axis. The unit normal in the plane $y_3 = 0$ at a point of the curve AB is

$$\mathbf{n} = n_{\alpha}\mathbf{E}^{\alpha}, \tag{5·1}$$

so that, from (2·12), the stress vector $\mathbf{t}$ lies in the plane $y_3 = 0$ and is given by

$$\mathbf{t} = \frac{n_{\alpha}\mathbf{T}_{\alpha}}{\surd A}. \tag{5·2}$$

Also
$$n_{\alpha} = \epsilon_{\alpha\beta}\frac{\mathrm{d}\theta^{\beta}}{\mathrm{d}s}. \tag{5·3}$$

The total force $\mathbf{P}$ exerted by the region 1 on the region 2 across a part AP of the curve AB, measured per unit length of the y_3-axis, is therefore, using (4·3), (4·4) and (4·9),

$$\begin{aligned}\mathbf{P} &= -\int_A^P \frac{n_\alpha \mathbf{T}_\alpha}{\sqrt{A}} \mathrm{d}s \\ &= -\int_A^P \epsilon_{\alpha\beta} \epsilon^{\gamma\alpha} \boldsymbol{\chi}_{,\gamma} \frac{\mathrm{d}\theta^\beta}{\mathrm{d}s} \mathrm{d}s \\ &= \int_A^P \boldsymbol{\chi}_{,\beta} \frac{\mathrm{d}\theta^\beta}{\mathrm{d}s} \mathrm{d}s = \boldsymbol{\chi} = \epsilon^{\rho\beta} \phi_{,\rho} \mathbf{E}_\beta, \end{aligned} \tag{5·4}$$

apart from an arbitrary constant vector which may be absorbed into $\boldsymbol{\chi}$ since this does not affect the stresses.

The total moment about the y_3-axis of the forces exerted by the region 1 on the region 2 across AP, measured per unit length of the y_3-axis, is

$$\mathbf{M} = \int_A^P [\mathbf{R} \wedge \boldsymbol{\chi}_{,\beta}] \frac{\mathrm{d}\theta^\beta}{\mathrm{d}s} \mathrm{d}s, \tag{5·5}$$

where

$$\mathbf{R} = R^\alpha \mathbf{E}_\alpha = R_\alpha \mathbf{E}^\alpha \tag{5·6}$$

is the position vector of a point on the curve with respect to the origin of the y_i-axes. By integrating (5·5) by parts we have

$$\mathbf{M} = [\mathbf{R} \wedge \boldsymbol{\chi}]_A^P - \int_A^P [\mathbf{E}_\alpha \wedge \boldsymbol{\chi}] \frac{\mathrm{d}\theta^\alpha}{\mathrm{d}s} \mathrm{d}s. \tag{5·7}$$

Now, from (4·4) and (5·6),

$$\begin{aligned}\mathbf{R} \wedge \boldsymbol{\chi} &= R^\alpha \chi^\beta \mathbf{E}_\alpha \wedge \mathbf{E}_\beta \\ &= R^\alpha \chi^\beta \epsilon_{\alpha\beta} \mathbf{E}^3 \\ &= R^\alpha \epsilon^{\rho\beta} \epsilon_{\alpha\beta} \phi_{,\rho} \mathbf{E}^3 \\ &= R^\alpha \phi_{,\alpha} \mathbf{E}^3,\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}_\alpha \wedge \boldsymbol{\chi} &= \chi^\beta \mathbf{E}_\alpha \wedge \mathbf{E}_\beta \\ &= \epsilon^{\rho\beta} \epsilon_{\alpha\beta} \phi_{,\rho} \mathbf{E}^3 = \phi_{,\alpha} \mathbf{E}^3.\end{aligned}$$

Hence, apart from an arbitrary constant which may be absorbed into ϕ without affecting the stresses,

$$\mathbf{M} = (R^\alpha \phi_{,\alpha} - \phi)\, \mathbf{E}^3, \tag{5·8}$$

and is therefore a couple of magnitude

$$M = R^\alpha \phi_{,\alpha} - \phi \tag{5·9}$$

about the y_3-axis.

If the curve AB is a bounding curve of a body and this curve is entirely free from applied forces, then

$$\boldsymbol{\chi} = 0 \tag{5·10}$$

at all points of this curve. From (4·4) and (4·9) we see that $\boldsymbol{\chi} = 0$ gives

$$\phi_{,1} = 0, \quad \phi_{,2} = 0 \tag{5·11}$$

at all points of AB, and this implies that ϕ is constant on AB. It follows from (5·9) that the couple M is zero for all arcs AP along AB if we remember that (5·9) is adjustable to the extent of an arbitrary constant. If we have a single boundary curve which is free from applied stress, ϕ may be taken to be zero on this boundary.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

EXACT SOLUTIONS

6. Flexure of cuboid and cylinder

A number of problems which have been solved by Rivlin (1949 *b*, *c*) for incompressible isotropic bodies can be considered as special cases of plane strain superposed on a uniform extension. It is instructive to examine one of Rivlin's problems from the point of view of the present paper. For this purpose we consider the flexure of a cuboid and we then give the solution of a generalized version of this problem in which a part of a thick cylindrical shell is deformed by flexure into part of another cylindrical shell.

Rivlin (1949 *b*, *c*) considers a cuboid bounded by planes $x_1 = a_1$, $x_1 = a_2$; $x_2 = \pm b$; $x_3 = \pm c$, where $a_1 - a_2 = 2a$. The cuboid is deformed symmetrically with respect to the x_1-axis so that:

(i) each plane initially normal to the x_1-axis becomes, in the deformed state, a portion of the curved surface of a cylinder whose axis is the x_3-axis;

(ii) planes initially normal to the x_2-axis become in the deformed state planes containing the x_3-axis;

(iii) there is a uniform extension λ in the direction of the x_3-axis.

Using the notation of § 3 we take cylindrical polar co-ordinates (r, θ, y_3) to define the strained body and we identify the curvilinear system θ_i with these co-ordinates so that

$$\left.\begin{aligned} \theta_1 = r, \quad \theta_2 = \theta, \quad \theta_3 = y_3, \\ y_1 = r\cos\theta, \quad y_2 = r\sin\theta. \end{aligned}\right\} \tag{6·1}$$

Also, in view of the assumptions (i), (ii) and (iii),

$$x_1 = f(r), \quad x_2 = g(\theta), \tag{6·2}$$

if the x_i- and y_i-axes coincide. Hence

$$A_{\alpha\beta} = \begin{pmatrix} 1, & 0 \\ 0, & r^2 \end{pmatrix}, \quad A^{\alpha\beta} = \begin{pmatrix} 1, & 0 \\ 0, & \dfrac{1}{r^2} \end{pmatrix}, \quad A = r^2, \quad a_{\alpha\beta} = \begin{pmatrix} f'^2, & 0 \\ 0, & g'^2 \end{pmatrix}, \tag{6·3}$$

and because of the incompressibility condition (3·19), $f'(r)\, g'(\theta) = \lambda r$, which leads, as shown by Rivlin, to

$$f(r) = \tfrac{1}{2}Dr^2 + B, \quad g(\theta) = \frac{\lambda\theta}{D}, \tag{6·4}$$

where
$$D = \frac{4a}{r_1^2 - r_2^2}, \quad B = \frac{a_2 r_1^2 - a_1 r_2^2}{r_1^2 - r_2^2}, \tag{6·5}$$

and r_1, r_2 $(r_1 > r_2)$ are the radii of the curved surfaces of the deformed body, which are initially the planes $x_1 = a_1$, $x_1 = a_2$ respectively. Thus, with the help of (6·4),

$$a_{\alpha\beta} = \begin{pmatrix} D^2r^2, & 0 \\ 0, & \dfrac{\lambda^2}{D^2} \end{pmatrix}, \quad a^{\alpha\beta} = \begin{pmatrix} \dfrac{1}{D^2r^2}, & 0 \\ 0, & \dfrac{D^2}{\lambda^2} \end{pmatrix}, \tag{6·6}$$

and, from (3·12),
$$I = \lambda^2 + \frac{D^2r^2}{\lambda^2} + \frac{1}{D^2r^2}. \tag{6·7}$$

When the body is incompressible the relevant equations of equilibrium are (4·15), and the only non-zero Christoffel symbols of the strained body are

$$\Gamma^1_{22} = -r, \quad \Gamma^2_{12} = \frac{1}{r}.$$

Putting $\alpha = 1$, $\beta = 2$ in (4·15) we find that ϕ depends only on r. Either of the remaining equations in (4·15) yield

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right) = H\left(\frac{D^2r}{\lambda^2} - \frac{1}{D^2r^3}\right). \tag{6·8}$$

But, from (6·7),

$$\frac{dI}{dr} = 2\left(\frac{D^2r}{\lambda^2} - \frac{1}{D^2r^3}\right),$$

hence, using also (3·20),

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d\phi}{dr}\right) = \frac{dW}{dI}\frac{dI}{dr} = \frac{dW}{dr},$$

which gives

$$\frac{1}{r}\frac{d\phi}{dr} = W(r) + C, \tag{6·9}$$

where C is a constant.

The resultant normal stress applied over the ends of the deformed cuboid can be found by using (4·17), and if this resultant stress is zero we then have an equation for λ.

When the applied stresses over the curved surfaces of the deformed cylinder are zero, $d\phi/dr$ vanishes for $r = r_1$, $r = r_2$, so that, from (6·9),

$$-C = W(r_1) = W(r_2) = W_0 \quad \text{(say)}. \tag{6·10}$$

From (5·4) we have

$$\mathbf{P} = \epsilon^{\rho\beta}\phi_{,\rho}\mathbf{E}_\beta = \frac{1}{r}\frac{d\phi}{dr}\mathbf{E}_2,$$

so that the resultant force acting on each of the surfaces initially at $x_2 = \pm b$ is the difference in $\frac{1}{r}\frac{d\phi}{dr}$ as r changes from r_1 to r_2, which is zero when the curved surfaces are free from traction. On the other hand, from (5·9), the couple per unit length of the deformed cylinder acting on each of these surfaces is

$$M = \left[R^1\frac{d\phi}{dr} - \phi\right]_{r_2}^{r_1} = \phi_{r_2} - \phi_{r_1}, \tag{6·11}$$

since $d\phi/dr$ vanishes at r_1 and r_2. Thus, using (6·9) and (6·10),

$$M = \tfrac{1}{2}(r_1^2 - r_2^2)\,W_0 - \int_{r_2}^{r_1} rW\,dr. \tag{6·12}$$

Stress components may be evaluated from (4·10) and (6·9), but we leave the discussion at this point.

7. Flexure of initially curved cuboid

We suppose that the *deformed* body is the same as that described in the previous section so that (with the same notation) it consists of part of a cylindrical shell bounded by the curved surfaces $r = r_1$, $r = r_2$ $(r_1 > r_2)$; the planes $\theta = \pm\alpha$, where α is a constant, and the planes $y_3 = \pm\lambda c$. In the undeformed body the surfaces corresponding to $r = r_1$, $r = r_2$ are concentric cylindrical surfaces of radii k_1, k_2 respectively; the surfaces corresponding to

$\theta = \pm\alpha$ are planes through the axis of these concentric cylinders; and the planes $y_3 = \pm\lambda c$ were originally given by $x_3 = \pm c$. The co-ordinates of the undeformed body are given by

$$\left.\begin{aligned} x_1 &= -k + F(r)\cos\phi(\theta), \\ x_2 &= F(r)\sin\phi(\theta), \end{aligned}\right\} \tag{7·1}$$

where the origins of the x_i- and y_i-axes are separated by a distance k but the x_1- and y_1-axes coincide.

The metric tensors of the deformed body are still given by (6·3), and, by using the incompressibility condition, we now find that

$$\phi(\theta) = \frac{\lambda\theta}{D_1}, \quad F(r) = \pm\surd(D_1 r^2 + B_1), \tag{7·2}$$

where
$$D_1 = \frac{k_1^2 - k_2^2}{r_1^2 - r_2^2}, \quad B_1 = \frac{k_2^2 r_1^2 - k_1^2 r_2^2}{r_1^2 - r_2^2}, \quad k_1 - k_2 = 2a. \tag{7·3}$$

If $k_1 > k_2 > 0$ the curvature of the cylindrical surfaces of the undeformed body has the same sign as the curvature of the cylindrical surfaces of the deformed body; but if $k_2 < k_1 < 0$ the undeformed and deformed cylindrical surfaces are curved in opposite directions. In (7·3) $2a$ is the thickness of the undeformed cylindrical shell. The upper sign in (7·2) in the expression for $F(r)$ corresponds to the case $k_1 > k_2 > 0$, whilst the lower sign corresponds to $k_2 < k_1 < 0$. We suppose that the planes $\theta = \pm\alpha$ in the deformed body correspond to the planes $\phi = \pm\beta$ in the undeformed body when $k_1 > k_2 > 0$, and to the planes $\phi = \mp\beta$ in the undeformed body when $k_2 < k_1 < 0$, so that, for both cases,

$$\lambda\alpha(r_1^2 - r_2^2) = \beta\,|\,k_1^2 - k_2^2\,|. \tag{7·4}$$

When $\beta = \pi$ and $k_2 < k_1 < 0$, the undeformed body is a complete cylindrical shell which, after being cut along a plane through its axis, is then deformed into a part of another cylindrical shell. The deformed shell is complete if $\alpha = \pi$. On the other hand, when $\alpha = \beta = \pi$ and $k_1 > k_2 > 0$, $D_1 = \lambda$, and we have the problem of a symmetrical inflation of a cylindrical tube.

The special case considered in § 6 in which the undeformed body was a cuboid can be obtained from the above by a limiting process. Thus, putting

$$k_2 = k + a_2, \quad k_1 = k + a_1, \tag{7·5}$$

and letting $k \to \infty$ whilst a_1, a_2 remain fixed we see from (7·3) that

$$D_1 \to kD, \quad B_1 \to k^2 + 2kB, \tag{7·6}$$

where D, B are given by (6·5). Hence, from (7·2),

$$\phi(\theta) \to \frac{\lambda\theta}{kD}, \quad F(r) \to k + f(r), \tag{7·7}$$

where $f(r)$ is given by (6·4), and equation (7·1) then gives

$$x_2 \to \frac{\lambda\theta}{D}, \quad x_1 \to f(r),$$

in agreement with (6·2).

Returning to the general case (7·1), where $F(r)$, $\phi(\theta)$ are given by (7·2), a straightforward calculation yields

$$a_{\alpha\beta} = \begin{pmatrix} \dfrac{D_1^2 r^2}{D_1 r^2 + B_1}, & 0 \\ 0, & \dfrac{\lambda^2(D_1 r^2 + B_1)}{D_1^2} \end{pmatrix}, \tag{7·8}$$

$$a^{\alpha\beta} = \begin{pmatrix} \dfrac{D_1 r^2 + B_1}{D_1^2 r^2}, & 0 \\ 0, & \dfrac{D_1^2}{\lambda^2(D_1 r^2 + B_1)} \end{pmatrix}, \tag{7·9}$$

and hence

$$I = \lambda^2 + \frac{D_1 r^2 + B_1}{D_1^2 r^2} + \frac{D_1^2 r^2}{\lambda^2(D_1 r^2 + B_1)}. \tag{7·10}$$

The equation for ϕ which corresponds to (6·8) is obtained from (4·15) in the form

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = H\left\{\frac{D_1^2 r}{\lambda^2(D_1 r^2 + B_1)} - \frac{D_1 r^2 + B_1}{D_1^2 r^3}\right\}, \tag{7·11}$$

the remaining equation being satisfied if ϕ is a function of r only. From (7·10)

$$\frac{\mathrm{d}I}{\mathrm{d}r} = \frac{2B_1}{D_1 r^2 + B_1}\left\{\frac{D_1^2 r}{\lambda^2(D_1 r^2 + B_1)} - \frac{D_1 r^2 + B_1}{D_1^2 r^3}\right\},$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = \frac{D_1 r^2 + B_1}{B_1}\frac{\mathrm{d}W}{\mathrm{d}r},$$

and therefore

$$\frac{1}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r} = W(r) + \frac{D_1}{B_1}\int_{r_2}^{r} r^2 \frac{\mathrm{d}W}{\mathrm{d}r}\,\mathrm{d}r + F, \tag{7·12}$$

where F is an arbitrary constant. The traction on each curved surface $r = r_1$, $r = r_2$ vanishes if $\mathrm{d}\phi/\mathrm{d}r$ is zero on these surfaces so that

$$\left.\begin{aligned} W(r_2) + F &= 0, \\ W(r_1) + \frac{D_1}{B_1}\int_{r_2}^{r_1} r^2 \frac{\mathrm{d}W}{\mathrm{d}r}\,\mathrm{d}r + F &= 0, \end{aligned}\right\} \tag{7·13}$$

and hence r_1, r_2 must be connected by the relation

$$W(r_2) - W(r_1) = \frac{D_1}{B_1}\int_{r_2}^{r_1} r^2 \frac{\mathrm{d}W}{\mathrm{d}r}\,\mathrm{d}r. \tag{7·14}$$

If equations (7·13) are satisfied then the couple M per unit length of the deformed cylinder acting on each end $\theta = \pm\alpha$ of the cylinder is, from (6·11),

$$\begin{aligned} M &= \phi_{r_2} - \phi_{r_1} \\ &= \tfrac{1}{2}(r_1^2 - r_2^2)\,W(r_2) - \int_{r_2}^{r_1} rW\,\mathrm{d}r - \frac{D_1}{B_1}\int_{r_2}^{r_1} r\,\mathrm{d}r \int_{r_2}^{r} x^2 \frac{\mathrm{d}W}{\mathrm{d}x}\,\mathrm{d}x, \end{aligned}$$

or

$$\begin{aligned} M &= \tfrac{1}{2}(r_1^2 - r_2^2)\,W(r_2) - \int_{r_2}^{r_1} rW\,\mathrm{d}r + \frac{D_1}{2B_1}\int_{r_2}^{r_1} r^2(r^2 - r_1^2)\,\frac{\mathrm{d}W}{\mathrm{d}r}\,\mathrm{d}r \\ &= \tfrac{1}{2}(r_1^2 - r_2^2)\left\{1 + \frac{D_1 r_2^2}{B_1}\right\} W(r_2) - \int_{r_2}^{r_1} \{1 + (D_1/B_1)\,(2r^2 - r_1^2)\}\,rW(r)\,\mathrm{d}r. \end{aligned} \tag{7·15}$$

In the special case when the undeformed body is a cuboid, we see, from (7·6), that $D_1/B_1 \to 0$, and we recover the formula (6·12) for M.

Further details of the stress distribution may be found from (4·10), (7·12) and (7·13), but we leave the discussion at this point.

8. Generalized shear

We consider the deformation of a cuboid which is initially bounded by the faces $x_1 = \pm a$, $x_2 = \pm b$, $x_3 = \pm c$, where a, b, c are constants, and we examine the possibility of a generalized shear in which each point moves parallel to the x_1-direction. If we take x_i-axes to coincide with y_i-axes, then we assume that

$$x_1 = y_1 + f(y_2), \quad x_2 = y_2, \quad x_3 = y_3,$$

and we identify our moving co-ordinate system θ_i with y_i. Hence, using the notation of § 3,

$$\left.\begin{aligned} A_{\alpha\beta} = A^{\alpha\beta} = \delta_{\alpha\beta}, \quad A = 1, \quad a = 1, \\ a_{\alpha\beta} = \begin{pmatrix} 1, & f' \\ f', & 1+f'^2 \end{pmatrix}, \quad a^{\alpha\beta} = \begin{pmatrix} 1+f'^2, & -f' \\ -f', & 1 \end{pmatrix}, \end{aligned}\right\} \tag{8·1}$$

where a dash on f denotes derivative with respect to $\theta_2 = y_2$. Since $a = A = 1$ the incompressibility condition is satisfied. From (3·6), since $\lambda = 1$,

$$I = 1 + a^{11} + a^{22} = 3 + f'^2. \tag{8·2}$$

With the help of (8·1) and (8·2) equations (4·15) for ϕ reduce to the two equations

$$\frac{\partial^2\phi}{\partial y_1^2} - \frac{\partial^2\phi}{\partial y_2^2} = -f'^2 H, \quad \frac{\partial^2\phi}{\partial y_1 \partial y_2} = f'H, \tag{8·3}$$

where we recall that H is given by (3·20) and is a function of $f'(y_2)$. Equations (8·3) are compatible if

$$\frac{d^2}{dy_2^2}(f'H) = 0,$$

or

$$f'H = A + By_2. \tag{8·4}$$

When $f' =$ constant, which corresponds to simple shear, H is a constant and (8·4) is satisfied with $B = 0$. The solution may then be completed in general terms (see Rivlin 1948 *d*). Otherwise, for a general form of strain energy, and therefore a general form of H, equation (8·4) becomes a differential equation for $f(y_2)$ which may be solved in the usual manner. This solution is particularly simple when the strain energy takes the approximate Mooney form (3·23), for then $H = 2(C_1 + C_2)$ and (8·4) becomes

$$2(C_1 + C_2) f' = A + By_2,$$

or

$$f = ky_2 + \tfrac{1}{2}k'y_2^2, \tag{8·5}$$

where k and k' are constants. It follows from (8·3) that

$$\frac{\partial^2\phi}{\partial y_1^2} - \frac{\partial^2\phi}{\partial y_2^2} = -2(C_1 + C_2)(k + k'y_2)^2,$$

$$\frac{\partial^2\phi}{\partial y_1 \partial y_2} = 2(C_1 + C_2)(k + k'y_2),$$

and hence $\phi = 2(C_1+C_2)\{(ky_2+\frac{1}{2}k'y_2^2)\,y_1+\frac{1}{6}k'y_1^3+\frac{1}{12}(6k^2y_2^2+4kk'y_2^3+k'^2y_2^4)\}$. (8·6)

In addition to (8·6) ϕ also contains terms of the form

$$a'(y_1^2+y_2^2)+b'y_1+c'y_2+d', \tag{8·7}$$

where a', b', c', d' are constants. The first two terms in (8·7) represent a uniform hydrostatic pressure and the linear terms contribute nothing to the stresses. We may determine the constant hydrostatic pressure by imposing a prescribed value for the term independent of y_1, y_2 in τ^{33} as given by (4·20).

Returning to the case of a general strain-energy function we see from (8·3) and (8·4) that ϕ has the form

$$\phi = y_1(Ay_2+\tfrac{1}{2}By_2^2)+\tfrac{1}{6}By_1^3+\psi(y_2), \tag{8·8}$$

apart from terms of the type (8·7), where $\psi(y_2)$ is a function of y_2 only determined by the equation

$$\frac{d^2\psi}{dy_2^2} = (A+By_2)\,f',$$

so that

$$\frac{d\psi}{dy_2} = (A+By_2)\,f-B\int f\,dy_2. \tag{8·9}$$

The solution may be completed when f is determined from (8·4).

It is also of interest to notice that (8·4) may be written

$$(A+By_2)\,f'' = 2f'f''\frac{dW}{dI} = \frac{dW}{dy_2}$$

if we use (3·20) and (8·2), so that

$$(A+By_2)\,f'-Bf = W+C, \tag{8·10}$$

where C is a constant. This equation may be integrated in the form

$$f = (A+By_2)\int\frac{W+C}{(A+By_2)^2}\,dy_2. \tag{8·11}$$

It appears, however, that in most cases it is easier to determine f from (8·4).

APPROXIMATION METHODS

9. Complex co-ordinates in deformed body

The use of complex variables has greatly simplified the formulation and solution of two-dimensional problems in classical linear elasticity, and it is natural to inquire whether complex variables may be used for problems of finite plane strain. In the case of finite deformations, however, we may choose complex co-ordinates in either the undeformed or the deformed body and we consider the latter case in this section.

We take the y_α-axes to coincide with the x_α-axes and put

$$z = y_1+iy_2, \quad \bar{z} = y_1-iy_2, \tag{9·1}$$

and we denote covariant and contravariant base vectors in the system of complex co-ordinates $(z, \bar{z})$ by $\mathbf{A}_\alpha$ and $\mathbf{A}^\alpha$ respectively. The position vector $\mathbf{R}$ of a point of the deformed body, which is given by (5·6), may then be written

$$\mathbf{R} = z^\alpha\mathbf{A}_\alpha = z_\alpha\mathbf{A}^\alpha. \tag{9·2}$$

By tensor transformations

$$z^1 = \frac{\partial z}{\partial y_1} y_1 + \frac{\partial z}{\partial y_2} y_2 = y_1 + \mathrm{i} y_2 = z,$$

$$z^2 = \frac{\partial \bar{z}}{\partial y_1} y_1 + \frac{\partial \bar{z}}{\partial y_2} y_2 = y_1 - \mathrm{i} y_2 = \bar{z},$$

so that complex co-ordinates $(z, \bar{z})$ may also be denoted by z^α.

We now take the moving system of co-ordinates θ_α to coincide with the system of complex co-ordinates $(z, \bar{z})$ so that

$$\theta_1 = z^1 = z, \quad \theta_2 = z^2 = \bar{z}. \tag{9·3}$$

The metric tensors $A_{\alpha\beta}$, $A^{\alpha\beta}$ then have the values

$$\left.\begin{aligned} A_{12} = \tfrac{1}{2}, \quad A_{11} = A_{22} = 0, \quad \sqrt{A} = \tfrac{1}{2}\mathrm{i} \\ A^{12} = 2, \quad A^{11} = A^{22} = 0. \end{aligned}\right\} \tag{9·4}$$

If the components of displacement along the x_α-axes are (u, v) then

$$y_1 = x_1 + u, \quad y_2 = x_2 + v,$$

and hence

$$\left.\begin{aligned} x_1 + \mathrm{i}x_2 = z - D, \\ x_1 - \mathrm{i}x_2 = \bar{z} - \bar{D}, \end{aligned}\right\} \tag{9·5}$$

where

$$D = u + \mathrm{i}v, \quad \bar{D} = u - \mathrm{i}v. \tag{9·6}$$

If the body is incompressible then, from (3·19) and (9·4),

$$\sqrt{a} = \lambda\sqrt{A} = \tfrac{1}{2}\mathrm{i}\lambda, \tag{9·7}$$

and

$$\sqrt{a} = \frac{\partial(x_1, x_2)}{\partial(\theta_1, \theta_2)} = \frac{\mathrm{i}}{2}\frac{\partial(x_1 + \mathrm{i}x_2, x_1 - \mathrm{i}x_2)}{\partial(z, \bar{z})}$$

$$= \frac{\mathrm{i}}{2}\left\{\left(1 - \frac{\partial D}{\partial z}\right)\left(1 - \frac{\partial \bar{D}}{\partial \bar{z}}\right) - \frac{\partial D}{\partial \bar{z}}\frac{\partial \bar{D}}{\partial z}\right\}.$$

Hence

$$\frac{\partial D}{\partial z} + \frac{\partial \bar{D}}{\partial \bar{z}} - \frac{\partial D}{\partial z}\frac{\partial \bar{D}}{\partial \bar{z}} + \frac{\partial D}{\partial \bar{z}}\frac{\partial \bar{D}}{\partial z} = 1 - \lambda. \tag{9·8}$$

Also

$$\left.\begin{aligned} a_{11} &= \left(\frac{\partial x_1}{\partial z}\right)^2 + \left(\frac{\partial x_2}{\partial z}\right)^2 = \left\{\frac{\partial(x_1 + \mathrm{i}x_2)}{\partial z}\right\}\left\{\frac{\partial(x_1 - \mathrm{i}x_2)}{\partial z}\right\} \\ &= \frac{\partial \bar{D}}{\partial z}\left(\frac{\partial D}{\partial z} - 1\right), \\ a_{22} &= \frac{\partial D}{\partial \bar{z}}\left(\frac{\partial \bar{D}}{\partial \bar{z}} - 1\right), \\ a_{12} &= \frac{1}{2}\left\{1 - \frac{\partial D}{\partial z} - \frac{\partial \bar{D}}{\partial \bar{z}} + \frac{\partial D}{\partial z}\frac{\partial \bar{D}}{\partial \bar{z}} + \frac{\partial D}{\partial \bar{z}}\frac{\partial \bar{D}}{\partial z}\right\} \\ &= \tfrac{1}{2}\lambda + \frac{\partial D}{\partial \bar{z}}\frac{\partial \bar{D}}{\partial z}, \end{aligned}\right\} \tag{9·9}$$

if we use (9·8) in order to simplify a_{12}. It follows from (3·12), (9·4) and (9·9) that

$$I = \lambda^2 + a^{12} = \lambda^2 + 4a_{12}/\lambda^2$$

$$= \lambda^2 + \frac{2}{\lambda} + \frac{4}{\lambda^2}\frac{\partial D}{\partial \bar{z}}\frac{\partial \bar{D}}{\partial z}. \tag{9·10}$$

Since the components (9·4) of the metric tensor of the deformed body are constants, the corresponding Christoffel symbols are zero and therefore covariant differentiation in the deformed body reduces to partial differentiation. If we now put $\alpha = 1$, $\beta = 1$ in equation (4·15) we have

$$\lambda^2 \frac{\partial^2 \phi}{\partial z^2} = H a_{11}. \tag{9·11}$$

When $\alpha = 2$, $\beta = 2$ equation (4·15) reduces to the complex conjugate of equation (9·11), and when $\alpha = 1$, $\beta = 2$ the equation (4·15) is identically satisfied. From (9·9) and (9·11) we obtain

$$\lambda^2 \frac{\partial^2 \phi}{\partial z^2} = \frac{\partial \overline{D}}{\partial z}\left(\frac{\partial D}{\partial z} - 1\right) H. \tag{9·12}$$

The incompressibility condition (9·8), together with equation (9·12) and its complex conjugate, are the fundamental equations for the functions ϕ, D, $\overline{D}$.

Denoting the stress components referred to complex co-ordinates in the deformed body by $T^{\alpha\beta}$, we obtain from (4·10)

$$T^{11} = \overline{T}^{22} = -4 \frac{\partial^2 \phi}{\partial \bar{z}^2}, \quad T^{12} = 4 \frac{\partial^2 \phi}{\partial z \partial \bar{z}}, \tag{9·13}$$

where a bar over a function denotes the complex conjugate of that function.

If the resultant force $\mathbf{P}$ across any arc AP of a curve in the deformed body has components (X, Y) along the y_1-, y_2-axes respectively, then a simple tensor transformation gives

$$\mathbf{P} = (X + \mathrm{i}Y)\, \mathbf{A}_1 + (X - \mathrm{i}Y)\, \mathbf{A}_2 = P\mathbf{A}_1 + \overline{P}\mathbf{A}_2. \tag{9·14}$$

Then remembering (9·4) we may interpret equation (5·4) in complex co-ordinates to get

$$P = 2\mathrm{i} \frac{\partial \phi}{\partial \bar{z}}. \tag{9·15}$$

Also, from (5·9), (9·2) and (9·3), the couple about the origin is

$$M = z \frac{\partial \phi}{\partial z} + \bar{z} \frac{\partial \phi}{\partial \bar{z}} - \phi. \tag{9·16}$$

From (9·15), or directly from (5·11), we have at all points of a boundary curve which is entirely free from applied stress

$$\frac{\partial \phi}{\partial z} = 0, \tag{9·17}$$

together with the complex conjugate of this equation.

10. Complex co-ordinates in deformed body: successive approximations

We now restrict our attention to plane strain for which $\lambda = 1$ so that

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{\partial \overline{D}}{\partial z}\left(\frac{\partial D}{\partial z} - 1\right) H, \tag{10·1}$$

$$I = 3 + 4 \frac{\partial D}{\partial \bar{z}} \frac{\partial \overline{D}}{\partial z}, \quad H = 2 \frac{\mathrm{d}W(I)}{\mathrm{d}I}. \tag{10·2}$$

Results for classical elasticity are obtained from these equations by neglecting squares and products of the displacement D and its derivatives with respect to z and $\bar{z}$. Further approximations may be obtained based on the classical theory as a first approximation. We therefore put

$$D = \epsilon({}^0D) + \epsilon^2({}^1D) + \dots, \tag{10·3}$$

where ϵ is a characteristic real parameter in a given problem. From (10·2) and (10·3),

$$I = 3 + 4\epsilon^2 \left\{ \frac{\partial\, {}^0D}{\partial \bar{z}} \frac{\partial\, {}^0\overline{D}}{\partial z} + \dots \right\}, \tag{10·4}$$

so that

$$H = {}^0H + 4\epsilon^2({}^2H) \frac{\partial\, {}^0D}{\partial \bar{z}} \frac{\partial\, {}^0\overline{D}}{\partial z} + \dots, \tag{10·5}$$

where

$${}^0H = 2\frac{\mathrm{d}W(I)}{\mathrm{d}I}, \quad {}^2H = 2\frac{\mathrm{d}^2W(I)}{\mathrm{d}I^2} \quad (I = 3). \tag{10·6}$$

The coefficients ${}^0H$, ${}^2H$, ... are therefore constants. We observe that ${}^0H = \frac{1}{3}E$, where E is the value of Young's modulus for strains corresponding to classical theory. If we now put

$$\phi = {}^0H\epsilon\{{}^0\phi + \epsilon\, {}^1\phi + \dots\}, \tag{10·7}$$

then equation (10·1) becomes

$$\frac{\partial^2({}^0\phi)}{\partial z^2} + \epsilon \frac{\partial^2({}^1\phi)}{\partial z^2} + \dots$$
$$= \left(1 + k_2\epsilon^2 \frac{\partial\, {}^0D}{\partial \bar{z}} \frac{\partial\, {}^0\overline{D}}{\partial z} + \dots\right)\left(-1 + \epsilon\frac{\partial\, {}^0D}{\partial z} + \epsilon^2 \frac{\partial\, {}^1D}{\partial z} + \dots\right)\left(\frac{\partial\, {}^0\overline{D}}{\partial z} + \epsilon\frac{\partial\, {}^1\overline{D}}{\partial z} + \dots\right), \tag{10·8}$$

where $k_2 = 4\, {}^2H/{}^0H$. Also, with $\lambda = 1$, the incompressibility condition (9·8) gives

$$\frac{\partial\, {}^0D}{\partial z} + \frac{\partial\, {}^0\overline{D}}{\partial \bar{z}} + \epsilon\left(\frac{\partial\, {}^1D}{\partial z} + \frac{\partial\, {}^1\overline{D}}{\partial \bar{z}}\right) + \dots - \epsilon\left(\frac{\partial\, {}^0D}{\partial z} + \epsilon\frac{\partial\, {}^1D}{\partial z} + \dots\right)\left(\frac{\partial\, {}^0\overline{D}}{\partial \bar{z}} + \epsilon\frac{\partial\, {}^1\overline{D}}{\partial \bar{z}} + \dots\right)$$
$$+ \epsilon\left(\frac{\partial\, {}^0D}{\partial \bar{z}} + \epsilon\frac{\partial\, {}^1D}{\partial \bar{z}} + \dots\right)\left(\frac{\partial\, {}^0\overline{D}}{\partial z} + \epsilon\frac{\partial\, {}^1\overline{D}}{\partial z} + \dots\right) = 0. \tag{10·9}$$

On equating to zero the various coefficients of ϵ in (10·8) and (10·9) we have

$$\frac{\partial^2({}^0\phi)}{\partial z^2} + \frac{\partial\, {}^0\overline{D}}{\partial z} = 0, \quad \frac{\partial\, {}^0D}{\partial z} + \frac{\partial\, {}^0\overline{D}}{\partial \bar{z}} = 0, \tag{10·10}$$

$$\left.\begin{aligned} \frac{\partial^2({}^1\phi)}{\partial z^2} + \frac{\partial\, {}^1\overline{D}}{\partial z} &= \frac{\partial\, {}^0D}{\partial z}\frac{\partial\, {}^0\overline{D}}{\partial z}, \\ \frac{\partial\, {}^1D}{\partial z} + \frac{\partial\, {}^1\overline{D}}{\partial \bar{z}} &= \frac{\partial\, {}^0D}{\partial z}\frac{\partial\, {}^0\overline{D}}{\partial \bar{z}} - \frac{\partial\, {}^0\overline{D}}{\partial z}\frac{\partial\, {}^0D}{\partial \bar{z}} \\ &= -\left\{\frac{\partial\, {}^0D}{\partial z}\right\}^2 - \frac{\partial\, {}^0\overline{D}}{\partial z}\frac{\partial\, {}^0D}{\partial \bar{z}}. \end{aligned}\right\} \tag{10·11}$$

Similar equations may be obtained from coefficients of higher powers of ϵ, but we restrict our attention at present to equations (10·10) and (10·11) which may be regarded as first and second approximations respectively. The first approximation corresponds to the classical theory, and the equations for this may be integrated in terms of complex potential functions $\Omega(z)$, $\omega(z)$. Thus

$$\left.\begin{aligned} {}^0\phi &= z\overline{\Omega}(\bar{z}) + \bar{z}\Omega(z) + \omega(z) + \overline{\omega}(\bar{z}), \\ {}^0D &= \Omega(z) - z\overline{\Omega}'(\bar{z}) - \overline{\omega}'(\bar{z}). \end{aligned}\right\} \tag{10·12}$$

Using (10·12), equations (10·11) now become

$$\frac{\partial^2({}^1\phi)}{\partial z^2}+\frac{\partial\,{}^1\overline{D}}{\partial z}=-\{\Omega'(z)-\overline{\Omega}'(\bar{z})\}\{\bar{z}\Omega''(z)+\omega''(z)\}, \tag{10·13}$$

$$\frac{\partial\,{}^1D}{\partial z}+\frac{\partial\,{}^1\overline{D}}{\partial \bar{z}}=-\{\Omega'(z)-\overline{\Omega}'(\bar{z})\}^2-\{z\overline{\Omega}''(\bar{z})+\overline{\omega}''(\bar{z})\}\{\bar{z}\Omega''(z)+\omega''(z)\}. \tag{10·14}$$

The first of these equations may be integrated to give

$$\frac{\partial\,{}^1\phi}{\partial z}+{}^1\overline{D}=2\overline{\Delta}(\bar{z})+\frac{1}{2}\int^{\bar{z}}\{\overline{\Omega}'(\bar{z})\}^2\,\mathrm{d}\bar{z}-\int^{z}\Omega'(z)\,\omega''(z)\,\mathrm{d}z$$
$$+\overline{\Omega}'(\bar{z})\{\bar{z}\Omega'(z)+\omega'(z)-\overline{\Omega}(\bar{z})\}-\tfrac{1}{2}\bar{z}\{\Omega'(z)\}^2, \tag{10·15}$$

where $\overline{\Delta}(\bar{z})$ is an arbitrary function of $\bar{z}$, the integral term which is a function of $\bar{z}$ only being added for convenience. From (10·14) and (10·15) it follows that

$$2\frac{\partial^2({}^1\phi)}{\partial z\,\partial \bar{z}}=2\Delta'(z)+2\overline{\Delta}'(\bar{z})$$
$$+\overline{\Omega}''(\bar{z})\{\bar{z}\Omega'(z)+\omega'(z)-\overline{\Omega}(\bar{z})\}$$
$$+\Omega''(z)\{z\overline{\Omega}'(\bar{z})+\overline{\omega}'(\bar{z})-\Omega(z)\}$$
$$+\{z\overline{\Omega}''(\bar{z})+\overline{\omega}''(\bar{z})\}\{\bar{z}\Omega''(z)+\omega''(z)\},$$

and hence

$$\frac{\partial\,{}^1\phi}{\partial \bar{z}}=\Delta(z)+z\overline{\Delta}'(\bar{z})+\bar{\delta}'(\bar{z})+\frac{1}{2}\int^{z}\{\Omega'(z)\}^2\,\mathrm{d}z-\int^{\bar{z}}\overline{\Omega}'(\bar{z})\,\overline{\omega}''(\bar{z})\,\mathrm{d}\bar{z}+\tfrac{1}{2}\Gamma-\tfrac{1}{2}z\{\overline{\Omega}'(\bar{z})\}^2, \tag{10·16}$$

where
$$\Gamma(z,\bar{z})=\{z\overline{\Omega}''(\bar{z})+\overline{\omega}''(\bar{z})\}\{\bar{z}\Omega'(z)+\omega'(z)-\overline{\Omega}(\bar{z})\}$$
$$+\{\Omega'(z)+\overline{\Omega}'(\bar{z})\}\{z\overline{\Omega}'(\bar{z})+\overline{\omega}'(\bar{z})-\Omega(z)\}, \tag{10·17}$$

and $\bar{\delta}(\bar{z})$ is a further arbitrary function of $\bar{z}$. By integration of (10·16) we may obtain ${}^1\phi$, but this is not required in applications of the theory. From (10·15) and (10·16)

$${}^1D=\Delta(z)-z\overline{\Delta}'(\bar{z})-\bar{\delta}'(\bar{z})-\tfrac{1}{2}\Lambda, \tag{10·18}$$

where
$$\Lambda(z,\bar{z})=\{z\overline{\Omega}''(\bar{z})+\overline{\omega}''(\bar{z})\}\{\bar{z}\Omega'(z)+\omega'(z)-\overline{\Omega}(\bar{z})\}$$
$$-\{\Omega'(z)-\overline{\Omega}'(\bar{z})\}\{z\overline{\Omega}'(\bar{z})+\overline{\omega}'(\bar{z})-\Omega(z)\}. \tag{10·19}$$

By introducing the first of equations (10·12) and equation (10·16) into (9·15) we may now obtain an explicit expression in terms of the complex potential functions for the resultant force across a curve.

In subsequent sections we shall be concerned with problems which are non-dislocational in character and the complex potential functions $\Omega(z), \omega(z), \Delta(z)$ and $\delta(z)$ must then be chosen so that the stress and displacement components are single-valued. It follows that ${}^0D, {}^1D, \ldots$ and all their derivatives with respect to z and $\bar{z}$, and similarly the second and higher order derivatives of ${}^0\phi, {}^1\phi, \ldots$, if they exist, must be single-valued at interior points of the body so that, from (10·12),

$$[\Omega'(z)]_c=0,\quad [\omega''(z)]_c=0,\quad [\Omega(z)]_c=[\overline{\omega}'(\bar{z})]_c, \tag{10·20}$$

where $[\;]_c$ denotes the change in value of the function inside the brackets during a complete circuit of a contour C lying entirely within the deformed body. Using (10·20) we see from

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(10·18) and (10·16) that the terms involving $\Omega(z)$ and $\omega(z)$ in the expressions for ${}^1D$, $\partial^2({}^1\phi)/\partial z^2$ and $\partial^2({}^1\phi)/\partial z\,\partial\bar{z}$ do not change in value during a complete circuit of the curve C. The complex potentials arising in the second approximation must therefore satisfy the similar conditions

$$[\Delta'(z)]_c = 0, \quad [\delta''(z)]_c = 0, \quad [\Delta(z)]_c = [\bar{\delta}'(\bar{z})]_c. \tag{10·21}$$

If the resultant force on a contour C in the deformed body is zero we must also have $[\partial\,{}^0\phi/\partial z]_c = [\partial\,{}^1\phi/\partial z]_c = 0$, and the relations (10·20) and (10·21) must then be replaced by

$$\left.\begin{aligned}[\Omega(z)]_c = 0, \quad [\omega'(z)]_c = 0,\\ [\Delta(z)]_c = 0, \quad [\delta'(z)]_c = 0.\end{aligned}\right\} \tag{10·22}$$

For some problems it is convenient to remove the integral terms from (10·16). Replacing $\Delta(z)$ by $\Delta(z) - \frac{1}{2}\int^z \{\Omega'(z)\}^2\,dz$ and $\delta'(z)$ by $\delta'(z) + \int^z \Omega'(z)\,\omega''(z)\,dz$ in (10·16) and (10·18) we obtain

$$\frac{\partial\,{}^1\phi}{\partial\bar{z}} = \Delta(z) + z\bar{\Delta}'(\bar{z}) + \bar{\delta}'(\bar{z}) + \tfrac{1}{2}\Gamma - z\{\bar{\Omega}'(\bar{z})\}^2, \tag{10·23}$$

$${}^1D = \Delta(z) - z\bar{\Delta}'(\bar{z}) - \bar{\delta}'(\bar{z}) - \tfrac{1}{2}\Lambda + \tfrac{1}{2}z\{\bar{\Omega}'(\bar{z})\}^2 - \frac{1}{2}\int^z \{\Omega'(z)\}^2\,dz - \int^{\bar{z}} \bar{\Omega}'(\bar{z})\,\bar{\omega}''(\bar{z})\,d\bar{z}. \tag{10·24}$$

The conditions (10·21) for single-valued stresses and displacements, however, now become

$$\left.\begin{aligned}[\Delta'(z)]_c = 0, \quad [\delta''(z)]_c = 0,\\ [\Delta(z) - \bar{\delta}'(\bar{z})]_c = \left[\frac{1}{2}\int^z \{\Omega'(z)\}^2\,dz + \int^{\bar{z}} \bar{\Omega}'(\bar{z})\,\bar{\omega}''(\bar{z})\,d\bar{z}\right]_c.\end{aligned}\right\} \tag{10·25}$$

If, as occurs in some problems, the integral terms are single-valued, these conditions reduce to (10·21). If, further, the resultant force on the contour C is zero we may again use conditions (10·22).

11. Complex co-ordinates in undeformed body

For some problems it is more convenient to take complex co-ordinates in the undeformed body. In this case we put

$$\left.\begin{aligned}\zeta = x_1 + \mathrm{i}x_2 = \theta_1, \qquad \bar{\zeta} = x_1 - \mathrm{i}x_2 = \theta_2,\\ z = y_1 + \mathrm{i}y_2 = \zeta + D, \quad \bar{z} = y_1 - \mathrm{i}y_2 = \bar{\zeta} + \bar{D},\end{aligned}\right\} \tag{11·1}$$

where the x_α-axes and y_α-axes still coincide and D is given by (9·6). The metric tensors $a_{\alpha\beta}$, $a^{\alpha\beta}$ are therefore given by

$$\left.\begin{aligned}a_{12} = \tfrac{1}{2}, \quad a_{11} = a_{22} = 0, \quad \sqrt{a} = \tfrac{1}{2}\mathrm{i},\\ a^{12} = 2, \quad a^{11} = a^{22} = 0,\end{aligned}\right\} \tag{11·2}$$

and the incompressibility condition (3·19) becomes

$$\sqrt{A} = \mathrm{i}/(2\lambda), \tag{11·3}$$

where

$$\sqrt{A} = \frac{\partial(y_1, y_2)}{\partial(\theta_1, \theta_2)} = \frac{\mathrm{i}}{2}\frac{\partial(z, \bar{z})}{\partial(\zeta, \bar{\zeta})} = \frac{\mathrm{i}}{2}\left\{\left(1 + \frac{\partial D}{\partial\zeta}\right)\left(1 + \frac{\partial\bar{D}}{\partial\bar{\zeta}}\right) - \frac{\partial D}{\partial\bar{\zeta}}\frac{\partial\bar{D}}{\partial\zeta}\right\}.$$

Hence
$$\frac{\partial D}{\partial \zeta}+\frac{\partial \bar{D}}{\partial \bar{\zeta}}+\frac{\partial D}{\partial \zeta}\frac{\partial \bar{D}}{\partial \bar{\zeta}}-\frac{\partial D}{\partial \bar{\zeta}}\frac{\partial \bar{D}}{\partial \zeta}=\frac{1}{\lambda}-1. \tag{11·4}$$

The metric tensors $A_{\alpha\beta}$ and $A^{\alpha\beta}$ are now given by

$$\left.\begin{aligned} A_{11}&=\frac{\partial \bar{D}}{\partial \zeta}\left(\frac{\partial D}{\partial \zeta}+1\right), \quad A_{22}=\frac{\partial D}{\partial \bar{\zeta}}\left(\frac{\partial \bar{D}}{\partial \bar{\zeta}}+1\right),\\ A_{12}&=\frac{1}{2\lambda}+\frac{\partial D}{\partial \bar{\zeta}}\frac{\partial \bar{D}}{\partial \zeta},\\ A^{11}&=\bar{A}^{22}=-4\lambda^2 A_{22}, \quad A^{12}=4\lambda^2 A_{12}, \end{aligned}\right\} \tag{11·5}$$

in which (11·4) has been used to simplify A_{12}. From (3·12), (11·2) and (11·5) we have

$$\begin{aligned} I&=\lambda^2+4A_{12}\\ &=\lambda^2+\frac{2}{\lambda}+4\frac{\partial D}{\partial \bar{\zeta}}\frac{\partial \bar{D}}{\partial \zeta}. \end{aligned} \tag{11·6}$$

Again, using (11·2), (11·5) and (11·6), the equations of equilibrium (4·16) reduce to

$$2\lambda^2\phi\,\|_1^2=2\lambda^2A^{2\alpha}\phi\,\|_{\alpha 1}=A^{22}H, \tag{11·7}$$

together with the complex conjugate of this equation. From (11·3) we may obtain by differentiation relations of the form

$$A_{11}A_{22,1}+A_{11,1}A_{22}-2A_{12}A_{12,1}=0, \tag{11·8}$$

and when these are used to simplify the resulting expressions, (11·5) and (11·7) yield

$$2A_{12}\frac{\partial^2\phi}{\partial\zeta^2}-2A_{11}\frac{\partial^2\phi}{\partial\zeta\partial\bar{\zeta}}+\frac{\partial A_{11}}{\partial\bar{\zeta}}\frac{\partial\phi}{\partial\zeta}-\frac{\partial A_{11}}{\partial\zeta}\frac{\partial\phi}{\partial\bar{\zeta}}+\frac{A_{11}H}{\lambda^2}=0. \tag{11·9}$$

Bearing in mind the relations (11·5), equation (11·4) together with (11·9) and its complex conjugate are sufficient to determine the functions ϕ, D, $\bar{D}$. Moreover, denoting the complex stress components referred to co-ordinates in the undeformed body by $T'^{\alpha\beta}$, we have from (4·10)

$$T'^{11}=\bar{T}'^{22}=-4\lambda^2\phi\,\|_{22}, \quad T'^{12}=4\lambda^2\phi\,\|_{12}. \tag{11·10}$$

It should be emphasized that $T'^{\alpha\beta}$ denotes components of the stress tensor across curves in the deformed body which were originally defined by complex co-ordinates $(\zeta,\bar{\zeta})$ in the undeformed body.

In order to obtain the resultant force $\mathbf{P}$ across any arc AP of a curve in the deformed body, we must express (5·4) in the appropriate form. If $\mathbf{v}$ is the displacement vector and $\mathbf{e}_i$ and $\mathbf{e}^i$ are the covariant and contravariant base vectors in the unstrained body we may write

$$\left.\begin{aligned} \mathbf{v}&=v^i\mathbf{e}_i=v_i\mathbf{e}^i,\\ \mathbf{E}_i&=\mathbf{e}_i+\mathbf{v}_{,i}=\mathbf{e}_i+v^j\,|_i\,\mathbf{e}_j, \end{aligned}\right\} \tag{11·11}$$

where the single line denotes covariant differentiation with respect to the undeformed body. From (5·4) and (11·11) we now have

$$\mathbf{P}=\epsilon^{\rho\beta}\phi_{,\rho}(\mathbf{e}_\beta+v^\alpha\,|_\beta\,\mathbf{e}_\alpha). \tag{11·12}$$

If $\mathbf{P}$ has components (X, Y) along the x_1- and x_2-axes respectively we have, as before, by a simple tensor transformation,

$$\mathbf{P}=(X+\mathrm{i}Y)\,\mathbf{a}_1+(X-\mathrm{i}Y)\,\mathbf{a}_2=P\mathbf{a}_1+\bar{P}\mathbf{a}_2, \tag{11·13}$$

where $\mathbf{a}_\alpha$ and $\mathbf{a}^\alpha$ are now used to denote the covariant and contravariant base vectors in the complex co-ordinate system $(\zeta, \bar{\zeta})$. Hence interpreting (11·12) in terms of complex co-ordinates and employing (11·3) we obtain

$$P = 2\mathrm{i}\lambda\left\{\frac{\partial\phi}{\partial\bar{\zeta}} + \frac{\partial D}{\partial\zeta}\frac{\partial\phi}{\partial\bar{\zeta}} - \frac{\partial D}{\partial\bar{\zeta}}\frac{\partial\phi}{\partial\zeta}\right\}. \tag{11·14}$$

On a boundary which is completely free from applied forces, we therefore have

$$\frac{\partial\phi}{\partial\bar{\zeta}} + \frac{\partial D}{\partial\zeta}\frac{\partial\phi}{\partial\bar{\zeta}} - \frac{\partial D}{\partial\bar{\zeta}}\frac{\partial\phi}{\partial\zeta} = 0. \tag{11·15}$$

This relation, together with its complex conjugate, are again equivalent to

$$\frac{\partial\phi}{\partial\zeta} = 0, \tag{11·16}$$

but the form (11·15) is frequently more convenient in practice.

The expression (5·7) for the couple may be similarly transformed, and we have, using (11·11) and (11·4), for the moment per unit length of y_3-axis

$$M = \lambda\left\{(\zeta+D)\left(1+\frac{\partial\bar{D}}{\partial\bar{\zeta}}\right) - (\bar{\zeta}+\bar{D})\frac{\partial D}{\partial\bar{\zeta}}\right\}\frac{\partial\phi}{\partial\zeta} + \lambda\left\{(\bar{\zeta}+\bar{D})\left(1+\frac{\partial D}{\partial\zeta}\right) - (\zeta+D)\frac{\partial\bar{D}}{\partial\zeta}\right\}\frac{\partial\phi}{\partial\bar{\zeta}} - \phi. \tag{11·17}$$

12. Complex co-ordinates in undeformed body: successive approximations

As before we restrict our attention to plane strain for which $\lambda = 1$. Equations (11·4) and (11·9) may be solved by successive approximations in a similar manner to that used in § 10. It is, however, somewhat simpler to derive results directly from those of § 10. From (11·1) we have

$$\frac{\partial}{\partial\zeta} = \left(1+\frac{\partial D}{\partial\zeta}\right)\frac{\partial}{\partial z} + \frac{\partial\bar{D}}{\partial\zeta}\frac{\partial}{\partial\bar{z}},$$

$$\frac{\partial}{\partial\bar{\zeta}} = \frac{\partial D}{\partial\bar{\zeta}}\frac{\partial}{\partial z} + \left(1+\frac{\partial\bar{D}}{\partial\bar{\zeta}}\right)\frac{\partial}{\partial\bar{z}},$$

so that, solving for $\partial/\partial z$ and $\partial/\partial\bar{z}$ and using (11·4), we obtain

$$\left.\begin{aligned}\frac{1}{\lambda}\frac{\partial}{\partial z} &= \left(1+\frac{\partial\bar{D}}{\partial\bar{\zeta}}\right)\frac{\partial}{\partial\zeta} - \frac{\partial\bar{D}}{\partial\zeta}\frac{\partial}{\partial\bar{\zeta}},\\ \frac{1}{\lambda}\frac{\partial}{\partial\bar{z}} &= \left(1+\frac{\partial D}{\partial\zeta}\right)\frac{\partial}{\partial\bar{\zeta}} - \frac{\partial D}{\partial\bar{\zeta}}\frac{\partial}{\partial\zeta}.\end{aligned}\right\} \tag{12·1}$$

We observe that the formula (11·14) for the resultant force on an arc of a curve may be obtained at once from (9·15) and (12·1).

From (10·3) and (11·1)

$$z = \zeta + \epsilon\{^0D(z,\bar{z})\} + \epsilon^2\{^1D(z,\bar{z})\} + \ldots, \tag{12·2}$$

and if we express D in the form

$$D = \epsilon\{^0D'(\zeta,\bar{\zeta})\} + \epsilon^2\{^1D'(\zeta,\bar{\zeta})\} + \ldots, \tag{12·3}$$

we also have

$$z = \zeta + \epsilon\{^0D'(\zeta,\bar{\zeta})\} + \epsilon^2\{^1D'(\zeta,\bar{\zeta})\} + \ldots. \tag{12·4}$$

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It follows from (12·2) and (12·4) that

$$\left.\begin{aligned} {}^0D'(\zeta,\bar{\zeta}) &= {}^0D(\zeta,\bar{\zeta}), \\ {}^1D'(\zeta,\bar{\zeta}) &= {}^1D(\zeta,\bar{\zeta}) + {}^0D(\zeta,\bar{\zeta})\frac{\partial\,{}^0D(\zeta,\bar{\zeta})}{\partial\zeta} + {}^0\bar{D}(\zeta,\bar{\zeta})\frac{\partial\,{}^0D(\zeta,\bar{\zeta})}{\partial\bar{\zeta}}. \end{aligned}\right. \tag{12·5}$$

From (10·7) and (12·4) we have

$$\frac{\partial\phi}{\partial\bar{z}} = {}^0H\epsilon\left\{\frac{\partial\,{}^0\phi(\zeta,\bar{\zeta})}{\partial\bar{\zeta}} + \epsilon\left[\frac{\partial\,{}^1\phi(\zeta,\bar{\zeta})}{\partial\bar{\zeta}} + {}^0D(\zeta,\bar{\zeta})\frac{\partial^2\{{}^0\phi(\zeta,\bar{\zeta})\}}{\partial\zeta\,\partial\bar{\zeta}} + {}^0\bar{D}(\zeta,\bar{\zeta})\frac{\partial^2\{{}^0\phi(\zeta,\bar{\zeta})\}}{\partial\bar{\zeta}^2}\right] + \ldots\right\}, \tag{12·6}$$

and if we again use (10·7) we obtain

$$\left.\begin{aligned} \frac{\partial\,{}^0\phi(z,\bar{z})}{\partial\bar{z}} &= \frac{\partial\,{}^0\phi(\zeta,\bar{\zeta})}{\partial\bar{\zeta}}, \\ \frac{\partial\,{}^1\phi(z,\bar{z})}{\partial\bar{z}} &= \frac{\partial\,{}^1\phi(\zeta,\bar{\zeta})}{\partial\bar{\zeta}} + {}^0D(\zeta,\bar{\zeta})\frac{\partial^2\{{}^0\phi(\zeta,\bar{\zeta})\}}{\partial\zeta\,\partial\bar{\zeta}} + {}^0\bar{D}(\zeta,\bar{\zeta})\frac{\partial^2\{{}^0\phi(\zeta,\bar{\zeta})\}}{\partial\bar{\zeta}^2}. \end{aligned}\right\} \tag{12·7}$$

Using (10·12), the first equations in (12·5) and (12·7) yield

$$\left.\begin{aligned} {}^0D'(\zeta,\bar{\zeta}) &= \Omega(\zeta) - \zeta\bar{\Omega}'(\bar{\zeta}) - \bar{\omega}'(\bar{\zeta}), \\ \frac{\partial\,{}^0\phi(z,\bar{z})}{\partial\bar{z}} &= \Omega(\zeta) + \zeta\bar{\Omega}'(\bar{\zeta}) + \bar{\omega}'(\bar{\zeta}). \end{aligned}\right\} \tag{12·8}$$

Also, from (10·12), (10·16), (10·18) and the remaining equations in (12·5) and (12·7),

$${}^1D'(\zeta,\bar{\zeta}) = \Delta(\zeta) - \zeta\bar{\Delta}'(\bar{\zeta}) - \bar{\delta}'(\bar{\zeta}) + \tfrac{1}{2}\Lambda(\zeta,\bar{\zeta}), \tag{12·9}$$

$$\frac{\partial\,{}^1\phi(z,\bar{z})}{\partial\bar{z}} = \Delta(\zeta) + \zeta\bar{\Delta}'(\bar{\zeta}) + \bar{\delta}'(\bar{\zeta}) + \frac{1}{2}\int^{\zeta}\{\Omega'(\zeta)\}^2\,d\zeta - \int^{\bar{\zeta}}\bar{\Omega}'(\bar{\zeta})\,\bar{\omega}''(\bar{\zeta})\,d\bar{\zeta} - \tfrac{1}{2}\zeta\{\bar{\Omega}'(\bar{\zeta})\}^2 - \tfrac{1}{2}\Gamma(\zeta,\bar{\zeta}), \tag{12·10}$$

where $\Gamma(\zeta,\bar{\zeta})$ and $\Lambda(\zeta,\bar{\zeta})$ are obtained from (10·17) and (10·19) by replacing $z,\bar{z}$ by $\zeta,\bar{\zeta}$ respectively.

The conditions for single-valued stresses and displacements are again of the forms (10·20) and (10·21) with $(\zeta,\bar{\zeta})$ replacing $(z,\bar{z})$. The results (12·9) and (12·10) are sufficient for dealing with problems for which either the displacement components or the stresses are prescribed at a boundary in the undeformed body. For some problems, however, it is convenient to remove the integral terms from the expression (12·10) by a process similar to that used in §10. Thus, replacing $\Delta(\zeta)$ by $\Delta(\zeta) - \frac{1}{2}\int^{\zeta}\{\Omega'(\zeta)\}^2\,d\zeta$ and $\delta'(\zeta)$ by $\delta'(\zeta) + \int^{\zeta}\Omega'(\zeta)\,\omega''(\zeta)\,d\zeta$ we obtain

$$\frac{\partial\,{}^1\phi(z,\bar{z})}{\partial\bar{z}} = \Delta(\zeta) + \zeta\bar{\Delta}'(\bar{\zeta}) + \bar{\delta}'(\bar{\zeta}) - \zeta\{\bar{\Omega}'(\bar{\zeta})\}^2 - \tfrac{1}{2}\Gamma(\zeta,\bar{\zeta}), \tag{12·11}$$

$${}^1D'(\zeta,\bar{\zeta}) = \Delta(\zeta) - \zeta\bar{\Delta}'(\bar{\zeta}) - \bar{\delta}'(\bar{\zeta}) + \tfrac{1}{2}\Lambda(\zeta,\bar{\zeta}) + \tfrac{1}{2}\zeta\{\bar{\Omega}'(\bar{\zeta})\}^2 - \frac{1}{2}\int^{\zeta}\{\Omega'(\zeta)\}^2\,d\zeta - \int^{\bar{\zeta}}\bar{\Omega}'(\bar{\zeta})\,\bar{\omega}''(\bar{\zeta})\,d\bar{\zeta}. \tag{12·12}$$

The conditions for single-valued stresses and displacements are now, however, of the form (10·25) with $(\zeta,\bar{\zeta})$ replacing $(z,\bar{z})$.

The stress components referred to complex co-ordinates in the deformed body are, from (9·13) and (12·1), when $\lambda = 1$, given by

$$\left.\begin{aligned} T^{11} = \overline{T}^{22} &= -4\left(1+\frac{\partial D}{\partial \zeta}\right)\frac{\partial}{\partial \bar\zeta}\left(\frac{\partial \phi}{\partial \bar z}\right)+4\frac{\partial D}{\partial \bar\zeta}\frac{\partial}{\partial \zeta}\left(\frac{\partial \phi}{\partial \bar z}\right),\\ T^{12} &= 4\left(1+\frac{\partial \overline{D}}{\partial \bar\zeta}\right)\frac{\partial}{\partial \zeta}\left(\frac{\partial \phi}{\partial \bar z}\right)-4\frac{\partial \overline{D}}{\partial \zeta}\frac{\partial}{\partial \bar\zeta}\left(\frac{\partial \phi}{\partial \bar z}\right).\end{aligned}\right\} \tag{12·13}$$

Hence using (10·7), (12·3) and (12·5), equations (12·13) become

$$\left.\begin{aligned} T^{11} = \overline{T}^{22} &= -4\epsilon\,{}^0H\left\{\frac{\partial}{\partial \bar\zeta}\left(\frac{\partial\,{}^0\phi}{\partial \bar z}\right)+\epsilon\left[\frac{\partial}{\partial \bar\zeta}\left(\frac{\partial\,{}^1\phi}{\partial \bar z}\right)+\frac{\partial\,{}^0D}{\partial \zeta}\frac{\partial}{\partial \bar\zeta}\left(\frac{\partial\,{}^0\phi}{\partial \bar z}\right)-\frac{\partial\,{}^0D}{\partial \bar\zeta}\frac{\partial}{\partial \zeta}\left(\frac{\partial\,{}^0\phi}{\partial \bar z}\right)\right]+\ldots\right\},\\ T^{12} &= 4\epsilon\,{}^0H\left\{\frac{\partial}{\partial \zeta}\left(\frac{\partial\,{}^0\phi}{\partial \bar z}\right)+\epsilon\left[\frac{\partial}{\partial \zeta}\left(\frac{\partial\,{}^1\phi}{\partial \bar z}\right)+\frac{\partial\,{}^0\overline{D}}{\partial \bar\zeta}\frac{\partial}{\partial \zeta}\left(\frac{\partial\,{}^0\phi}{\partial \bar z}\right)-\frac{\partial\,{}^0\overline{D}}{\partial \zeta}\frac{\partial}{\partial \bar\zeta}\left(\frac{\partial\,{}^0\phi}{\partial \bar z}\right)\right]+\ldots\right\},\\ &= 4\epsilon\,{}^0H\left\{\frac{\partial}{\partial \bar\zeta}\left(\frac{\partial\,{}^0\phi}{\partial z}\right)+\epsilon\left[\frac{\partial}{\partial \bar\zeta}\left(\frac{\partial\,{}^1\phi}{\partial z}\right)+\frac{\partial\,{}^0D}{\partial \zeta}\frac{\partial}{\partial \bar\zeta}\left(\frac{\partial\,{}^0\phi}{\partial z}\right)-\frac{\partial\,{}^0D}{\partial \bar\zeta}\frac{\partial}{\partial \zeta}\left(\frac{\partial\,{}^0\phi}{\partial z}\right)\right]+\ldots\right\},\end{aligned}\right\} \tag{12·14}$$

the two forms of T^{12} being equivalent on account of (12·1) and (12·7).

13. Transformations of the stress and displacement components

It is sometimes convenient, in applications of the foregoing theory, to obtain the solution of a problem by the use of a given reference frame, but to express the final results in terms of a different system of co-ordinates. This may be achieved by means of simple tensor transformations, and we shall summarize the results required for subsequent sections. We recall that rectangular Cartesian co-ordinates in the undeformed and deformed bodies are denoted by x_α and y_α respectively, with corresponding complex co-ordinates $(\zeta, \bar\zeta)$ and $(z, \bar z)$, where

$$\left.\begin{aligned} \zeta &= \zeta^1 = x_1+\mathrm{i}x_2, & \bar\zeta &= \zeta^2 = x_1-\mathrm{i}x_2,\\ z &= z^1 = y_1+\mathrm{i}y_2, & \bar z &= z^2 = y_1-\mathrm{i}y_2,\\ z &= \zeta+D, & \bar z &= \bar\zeta+\overline{D}.\end{aligned}\right\} \tag{13·1}$$

The contravariant components of the stress tensor referred to the complex co-ordinates $(z, \bar z)$ in the deformed body have already been denoted by $T^{\alpha\beta}$. The corresponding (physical) stress components referred to the y_α-axes are $t^{\lambda\mu}$. The contravariant components of stress in the deformed body along curves which were originally defined by the rectangular Cartesian co-ordinates x_α in the undeformed body are denoted† by $t'^{\lambda\mu}$, and the corresponding contravariant components referred to the complex co-ordinates $(\zeta, \bar\zeta)$ by $T'^{\lambda\mu}$. By the use of transformations of the form

$$T'^{\alpha\beta} = \frac{\partial \zeta^\alpha}{\partial x_\lambda}\frac{\partial \zeta^\beta}{\partial x_\mu}t'^{\lambda\mu}, \quad T^{\alpha\beta} = \frac{\partial z^\alpha}{\partial y_\lambda}\frac{\partial z^\beta}{\partial y_\mu}t^{\lambda\mu}, \tag{13·2}$$

and the employment of (13·1) we obtain

$$\left.\begin{aligned} T'^{11} = \overline{T}'^{22} &= t'^{11}-t'^{22}+2\mathrm{i}t'^{12},\\ T'^{12} &= t'^{11}+t'^{22},\end{aligned}\right\} \tag{13·3}$$

† These are not physical components of stress.

and
$$\left.\begin{aligned} T^{11} = \overline{T}^{22} &= t^{11}-t^{22}+2\mathrm{i}t^{12}, \\ T^{12} &= t^{11}+t^{22}, \end{aligned}\right\} \tag{13·4}$$

from which we may derive the inverse relations

$$\left.\begin{aligned} 4t^{11} &= \quad T^{11}+T^{22}+2T^{12}, \\ 4t^{22} &= -T^{11}-T^{22}+2T^{12}, \\ 4\mathrm{i}t^{12} &= \quad T^{11}-T^{22}, \end{aligned}\right\} \tag{13·5}$$

with a similar set of equations corresponding to (13·3). Similarly, by the use of the last of equations (13·1) and the incompressibility condition in the forms (9·8) and (11·4) we may obtain

$$\left.\begin{aligned} T^{11} = \overline{T}^{22} &= \left(1+\frac{\partial D}{\partial \zeta}\right)^2 T'^{11}+\left(\frac{\partial D}{\partial \bar{\zeta}}\right)^2 T'^{22}+2\frac{\partial D}{\partial \bar{\zeta}}\left(1+\frac{\partial D}{\partial \zeta}\right)T'^{12}, \\ T^{12} &= \frac{\partial \overline{D}}{\partial \zeta}\left(1+\frac{\partial D}{\partial \zeta}\right)T'^{11}+\frac{\partial D}{\partial \bar{\zeta}}\left(1+\frac{\partial \overline{D}}{\partial \bar{\zeta}}\right)T'^{22}+\left(\frac{1}{\lambda}+2\frac{\partial \overline{D}}{\partial \zeta}\frac{\partial D}{\partial \bar{\zeta}}\right)T'^{12}, \end{aligned}\right\} \tag{13·6}$$

and
$$\left.\begin{aligned} T'^{11} = \overline{T}'^{22} &= \left(1-\frac{\partial D}{\partial z}\right)^2 T^{11}+\left(\frac{\partial D}{\partial \bar{z}}\right)^2 T^{22}-2\frac{\partial D}{\partial \bar{z}}\left(1-\frac{\partial D}{\partial z}\right)T^{12}, \\ T'^{12} &= -\frac{\partial \overline{D}}{\partial z}\left(1-\frac{\partial D}{\partial z}\right)T^{11}-\frac{\partial D}{\partial \bar{z}}\left(1-\frac{\partial \overline{D}}{\partial \bar{z}}\right)T^{22}+\left(\lambda+2\frac{\partial \overline{D}}{\partial z}\frac{\partial D}{\partial \bar{z}}\right)T^{12}. \end{aligned}\right\} \tag{13·7}$$

The equations for a rotation of the frame of reference assume a very simple form in complex variable notation. Let points in the undeformed body now be referred to a rectangular Cartesian co-ordinate system x^*_α which is such that the x^*_1-axis is inclined at an angle θ to the x_1-axis. Then writing $\zeta^* = x^*_1+\mathrm{i}x^*_2$ we have

$$\zeta^* = \zeta\,\mathrm{e}^{-\mathrm{i}\theta}, \quad \bar{\zeta}^* = \bar{\zeta}\,\mathrm{e}^{\mathrm{i}\theta},$$

and denoting the complex stress components in the system $(\zeta^*, \bar{\zeta}^*)$ by $T^{*\alpha\beta}$, a tensor transformation of the form (13·2) yields

$$T^{*11} = \overline{T}^{*22} = \mathrm{e}^{-2\mathrm{i}\theta}\,T'^{11}, \quad T^{*12} = T'^{12}. \tag{13·8}$$

Similarly, for the complex displacement D^* in the system $(\zeta^*, \bar{\zeta}^*)$ we have

$$D^* = \frac{\partial \zeta^*}{\partial \zeta}D = \mathrm{e}^{-\mathrm{i}\theta}D. \tag{13·9}$$

If points in the undeformed body are specified with respect to a polar co-ordinate system (r,θ) we may choose the axes (x^*_1, x^*_2) to coincide with the radial and tangential directions respectively at the point (r,θ), where $\zeta = r\,\mathrm{e}^{\mathrm{i}\theta}$. Denoting the contravariant components† of the stress tensor in the (r,θ) system by $t^{*\alpha\beta}$ and the physical displacement vector components by u_r, u_θ we have, from (13·8), (13·9) and equations of the type (13·3),

$$\left.\begin{aligned} t^{*11}-t^{*22}+2\mathrm{i}t^{*12} &= \mathrm{e}^{-2\mathrm{i}\theta}\,T'^{11}, \\ t^{*11}+t^{*22} &= T'^{12}, \\ u_r+\mathrm{i}u_\theta &= \mathrm{e}^{-\mathrm{i}\theta}D. \end{aligned}\right\} \tag{13·10}$$

† These are not physical components of stress.

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Similar results may be obtained for co-ordinate systems chosen with reference to the deformed body. If polar co-ordinates (r, θ) are chosen in the deformed body where $z = r\,\mathrm{e}^{\mathrm{i}\theta}$, then the equations corresponding to (13·10) give the actual *physical* components of stress and displacement in polar co-ordinates in terms of $T^{\alpha\beta}$ and D. Thus, with an obvious notation for physical components

$$\left.\begin{aligned} t_{rr}-t_{\theta\theta}+2\mathrm{i}t_{r\theta} &= \mathrm{e}^{-2\mathrm{i}\theta}\,T^{11},\\ t_{rr}+t_{\theta\theta} &= T^{12},\\ u_r+\mathrm{i}u_\theta &= \mathrm{e}^{-\mathrm{i}\theta}\,D.\end{aligned}\right\} \tag{13·11}$$

14. Infinite body containing a circular hole under a uniform tension at infinity

We shall now consider the problem of an infinite elastic body subjected to a uniform tension T in the direction of the x_1- (or y_1-) axis at infinity, and which contains a circular hole of radius a, the boundary of which is completely free from applied forces. The hole may either be assumed circular in the undeformed state, in which case the problem is conveniently treated by choosing complex co-ordinates in the unstrained body, or it may be assumed to have such a shape initially that it becomes circular after deformation, in which case complex co-ordinates in the deformed body become appropriate. The two problems have a number of features in common, but we shall consider the latter case first, since this is, in some respects, simpler.

Choosing complex co-ordinates $(z, \bar{z})$ in the deformed body, we may, without loss of generality, assume the centre of the hole to coincide with the origin of co-ordinates, so that the circular boundary may be described by the equation

$$z\bar{z} = a^2. \tag{14·1}$$

Since this boundary is free from applied stress, we have, from (9·17), the conditions

$$\frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial\bar{z}} = 0 \quad \text{on} \quad z\bar{z} = a^2. \tag{14·2}$$

Also, to ensure a uniform tension T in a direction parallel to the y_1-axis at infinity, we must have from (13·4), (9·13) and (9·15)

$$\frac{\partial^2\phi}{\partial z^2} = \frac{\partial^2\phi}{\partial\bar{z}^2} = -\frac{\partial^2\phi}{\partial z\,\partial\bar{z}} = -\tfrac{1}{4}T + O\left(\frac{1}{|z|^2}\right) \tag{14·3}$$

for large $|z|$, since the resultant force on any (large) circuit surrounding the circle is zero.†

We may now assume that D, I, H and ϕ can be expanded in the forms given by (10·3), (10·4), (10·5) and (10·7) respectively, and we shall take the real parameter ϵ to have the value $T/(4\,{}^0H) = 3T/(4E)$. From the boundary conditions (14·2) and (14·3) we then have

$$\frac{\partial({}^n\phi)}{\partial z} = \frac{\partial({}^n\phi)}{\partial\bar{z}} = 0 \quad (n = 0, 1, 2, \ldots) \quad \text{on} \quad z\bar{z} = a^2, \tag{14·4}$$

and

$$\left.\begin{aligned} \frac{\partial^2({}^0\phi)}{\partial z^2} = \frac{\partial^2({}^0\phi)}{\partial\bar{z}^2} = -\frac{\partial^2({}^0\phi)}{\partial z\,\partial\bar{z}} &= -1 + O\left(\frac{1}{|z|^2}\right),\\ \frac{\partial^2({}^n\phi)}{\partial z^2} = \frac{\partial^2({}^n\phi)}{\partial\bar{z}^2} = \frac{\partial^2({}^n\phi)}{\partial z\,\partial\bar{z}} &= O\left(\frac{1}{|z|^2}\right) \quad (n>0)\end{aligned}\right\} \tag{14·5}$$

† Equations (10·7), (10·12), (10·16) and (10·22) are also used to determine the order of magnitude of terms vanishing at infinity.

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for large $|z|$, where ${}^0\phi$ and ${}^1\phi$ may be expressed in terms of the complex potential functions $\Omega(z)$, $\omega(z)$, $\Delta(z)$ and $\delta(z)$ by the equations of § 10. Using (10·12), the boundary conditions for the first approximation become

$$\bar{\Omega}(\bar{z})+\bar{z}\Omega'(z)+\omega'(z)=0 \quad \text{on} \quad z\bar{z}=a^2, \tag{14·6}$$

and

$$\bar{\Omega}(\bar{z})+\bar{z}\Omega'(z)+\omega'(z)=\bar{z}-z+O\left(\frac{1}{|z|}\right), \tag{14·7}$$

for large $|z|$. Remembering the conditions (10·22), equations (14·7) will be satisfied if the potential functions $\Omega(z)$ and $\omega(z)$ have the forms

$$\left.\begin{aligned}\Omega(z) &= \tfrac{1}{2}z+\sum_{r=1}^{\infty} b_r z^{-r},\\ \omega(z) &= -\tfrac{1}{2}z^2+Az+B\log z+\sum_{r=1}^{\infty} c_r z^{-r},\end{aligned}\right\} \tag{14·8}$$

where b_r, c_r, A and B are constants. Introducing these expressions into (14·6), we find that this relation can only be satisfied if

$$b_1=a^2, \qquad b_r=0 \quad (r\neq 1),$$
$$c_2=-\tfrac{1}{2}a^4, \quad c_r=0 \quad (r\neq 2),$$
$$A=0, \quad B=-a^2,$$

and equations (14·8) then reduce to

$$\left.\begin{aligned}\Omega(z) &= \frac{1}{2}\left(z+\frac{2a^2}{z}\right),\\ \omega(z) &= -\frac{1}{2}\left(z^2+2a^2\log z+\frac{a^4}{z^2}\right).\end{aligned}\right\} \tag{14·9}$$

Since the boundary conditions are given in terms of the applied forces, and since the integral terms in (10·25) are here single-valued, it is convenient in obtaining the second approximation terms to use the forms for ${}^1\phi$ and ${}^1D$ given by (10·23) and (10·24) respectively. From (14·9) and (10·23) we have

$$\begin{aligned}\frac{\partial\, {}^1\phi}{\partial \bar{z}} &= \Delta(z)+z\bar{\Delta}'(\bar{z})+\bar{\delta}'(\bar{z})\\ &\quad +\frac{1}{2}\left\{\tfrac{1}{2}z-\bar{z}+a^2\left(\frac{1}{\bar{z}}+\frac{2\bar{z}}{z^2}-\frac{2z^2}{\bar{z}^3}\right)-a^4\left(\frac{1}{z\bar{z}^2}+\frac{1}{\bar{z}^3}\right)+a^6\left(\frac{2}{\bar{z}^5}+\frac{3}{z\bar{z}^4}+\frac{4}{z^2\bar{z}^3}+\frac{1}{z^3\bar{z}^2}\right)-\frac{3a^8}{z^3\bar{z}^4}\right\},\end{aligned} \tag{14·10}$$

and (14·4) and (14·5) then yield the conditions

$$\left.\begin{aligned}\Delta(z)+z\bar{\Delta}'(\bar{z})+\bar{\delta}'(\bar{z})+\frac{z^3}{a^2}+\frac{3z}{4}+\frac{a^4}{z^3} &= 0 \quad \text{on} \quad z\bar{z}=a^2,\\ \bar{\Delta}(\bar{z})+\bar{z}\Delta'(z)+\delta'(z) &= \frac{z}{2}-\frac{\bar{z}}{4}+O\left(\frac{1}{|z|}\right) \quad \text{as} \quad |z|\to\infty.\end{aligned}\right\} \tag{14·11}$$

The determination of complex potential functions which satisfy (14·11) and yield single-valued expressions for the stresses and displacements proceeds exactly as for the first approximation. We obtain

$$\left.\begin{aligned}\Delta(z) &= -\frac{1}{8}\left(z+\frac{4a^2}{z}+\frac{8a^4}{z^3}\right),\\ \delta(z) &= \frac{1}{4}\left(z^2-2a^2\log z+\frac{3a^4}{z^2}+\frac{3a^6}{z^4}\right).\end{aligned}\right\} \qquad (14\cdot12)$$

The stress functions ${}^0\phi$ and ${}^1\phi$ may now be found from (10·12), (14·9), (14·10) and (14·12), and introducing the resulting expressions into (9·13), with $\lambda = 1$, we find for the complex stress components referred to co-ordinates in the deformed body,

$$T^{11} = \overline{T}^{22} = T\left\{\left[1-a^2\left(\frac{2z}{\bar{z}^3}+\frac{1}{\bar{z}^2}\right)+\frac{3a^4}{\bar{z}^4}\right]\right.$$
$$\left.-\frac{3Ta^2}{4E}\left[\left(\frac{1}{z^2}-\frac{z}{\bar{z}^3}+\frac{3z^2}{\bar{z}^4}\right)+a^2\left(\frac{1}{z\bar{z}^3}+\frac{6}{\bar{z}^4}-\frac{12z}{\bar{z}^5}\right)+a^4\left(\frac{10}{\bar{z}^6}-\frac{6}{z\bar{z}^5}-\frac{6}{z^2\bar{z}^4}-\frac{1}{z^3\bar{z}^3}\right)+\frac{6a^6}{z^3\bar{z}^5}\right]\right\}, \qquad (14\cdot13)$$

$$T^{12} = T\left\{\left[1-a^2\left(\frac{1}{z^2}+\frac{1}{\bar{z}^2}\right)\right]\right.$$
$$\left.+\frac{3Ta^2}{8E}\left[\left(\frac{1}{z^2}+\frac{1}{\bar{z}^2}-\frac{4z}{\bar{z}^3}-\frac{4\bar{z}}{z^3}\right)+a^2\left(\frac{1}{z^2\bar{z}^2}+\frac{6}{z^4}+\frac{6}{\bar{z}^4}\right)-a^4\left(\frac{8}{z^3\bar{z}^3}+\frac{3}{z^4\bar{z}^2}+\frac{3}{z^2\bar{z}^4}\right)+\frac{9a^6}{z^4\bar{z}^4}\right]\right\}. \qquad (14\cdot14)$$

Now since from (13·4) T^{12} is the sum of the principal stresses, it is invariant under rotations of axes, and therefore, on the boundary of the hole $z\bar{z} = a^2$, on which the normal and shear stresses are zero, it gives the hoop stress. Introducing polar co-ordinates (r, θ) in the deformed body by means of the relations

$$y_1 = r\cos\theta, \quad y_2 = r\sin\theta, \qquad (14\cdot15)$$

we obtain from (14·14) for the hoop stress on $z\bar{z} = a^2$

$$[T^{12}]_{r=a} = T\left\{(1-2\cos 2\theta)+\frac{3T}{4E}(1-2\cos 2\theta+2\cos 4\theta)\right\}. \qquad (14\cdot16)$$

On the axes of symmetry this expression reduces to

$$\left.\begin{aligned}&[T^{12}]_{r=a} = -T\left\{1-\frac{3T}{4E}\right\} \quad \text{at} \quad \theta = 0 \quad \text{and} \quad \theta = \pi,\\ \text{and} \quad &[T^{12}]_{r=a} = 3T\left\{1+\frac{5T}{4E}\right\} \quad \text{at} \quad \theta = \tfrac{1}{2}\pi \quad \text{and} \quad \theta = \tfrac{3}{2}\pi.\end{aligned}\right\} \qquad (14\cdot17)$$

The complex displacement functions ${}^0D$ and ${}^1D$ are found by introducing (14·9) and (14·12) into the second of equations (10·12) and (10·24) respectively. We obtain

$${}^0D(z,\bar{z}) = \bar{z}+a^2\left(\frac{1}{z}+\frac{1}{\bar{z}}+\frac{z}{\bar{z}^2}\right)-\frac{a^4}{\bar{z}^3}, \qquad (14\cdot18)$$

$${}^1D(z,\bar{z}) = -\frac{1}{2}\left\{z+a^2\left(\frac{3}{z}-\frac{2}{\bar{z}}+\frac{z}{\bar{z}^2}-\frac{2z^2}{\bar{z}^3}\right)\right.$$
$$\left.-\tfrac{1}{3}a^4\left(\frac{1}{z^3}+\frac{10}{\bar{z}^3}+\frac{6}{z^2\bar{z}}+\frac{9}{z\bar{z}^2}-\frac{21z}{\bar{z}^4}\right)+\tfrac{1}{5}a^6\left(\frac{5}{z^3\bar{z}^2}+\frac{30}{z^2\bar{z}^3}+\frac{15}{z\bar{z}^4}-\frac{26}{\bar{z}^5}\right)-\frac{3a^8}{z^3\bar{z}^4}\right\}. \qquad (14\cdot19)$$

The real components of displacement in the co-ordinate system y_α may be obtained from these expressions in the usual manner by the use of (10·3), (9·1) and (9·6) and the separation of real and imaginary parts in the resulting equations. Alternatively, by making use of the last of equations (13·11) we may express the results in terms of the polar co-ordinate system (r, θ). At the boundary of the hole where $r = a$, the displacement components u_r, u_θ in the radial and tangential directions respectively assume the comparatively simple forms

$$\left.\begin{aligned}[u_r]_{r=a} &= \frac{3Ta}{4E}\left\{(1+2\cos 2\theta)+\frac{T}{40E}(15-40\cos 2\theta+8\cos 4\theta)\right\},\\ [u_\theta]_{r=a} &= -\frac{3Ta}{2E}\left\{1-\frac{T}{20E}(5-\cos 2\theta)\right\}\sin 2\theta.\end{aligned}\right\} \tag{14·20}$$

We shall now consider the case where the hole is initially circular, and for this purpose we choose complex co-ordinates $(\zeta, \bar{\zeta})$ in the undeformed body, and employ the notation and equations of §§ 11 and 12. It may be shown that the first approximation solution is unchanged apart from notation so that from (14·9)

$$\left.\begin{aligned}\Omega(\zeta) &= \frac{1}{2}\left(\zeta+\frac{2a^2}{\zeta}\right),\\ \omega(\zeta) &= -\frac{1}{2}\left(\zeta^2+2a^2\log\zeta+\frac{a^4}{\zeta^2}\right).\end{aligned}\right\} \tag{14·21}$$

To ensure a uniform tension of magnitude T in the direction of the x_1- (or y_1-) axis at infinity we again have from (13·4)

$$T^{11} = T^{22} = T^{12} = T, \tag{14·22}$$

or, in terms of complex co-ordinates in the deformed body,

$$\frac{\partial\phi}{\partial\bar{z}} = \frac{T}{4}(z-\bar{z}) = \epsilon\,{}^0H(z-\bar{z})+O\left(\frac{1}{|z|}\right) \tag{14·23}$$

for large $|z|$. Hence, from (12·4), (10·7) and (14·23),

$$\left.\begin{aligned}\frac{\partial\,{}^0\phi}{\partial\bar{z}} &= \zeta-\bar{\zeta},\\ \frac{\partial\,{}^1\phi}{\partial\bar{z}} &= {}^0D'(\zeta,\bar{\zeta})-{}^0\bar{D}'(\zeta,\bar{\zeta}),\end{aligned}\right\} \tag{14·24}$$

for large $|\zeta|$, and using (12·8) and (14·21) the second of equations (14·24) becomes

$$\frac{\partial\,{}^1\phi}{\partial\bar{z}} = \bar{\zeta}-\zeta \tag{14·25}$$

for large $|\zeta|$. Combining (14·25) with (14·21) and (12·11) we have for large $|\zeta|$

$$\Delta(\zeta)+\zeta\bar{\Delta}'(\bar{\zeta})+\bar{\delta}'(\bar{\zeta}) = \tfrac{1}{2}\bar{\zeta}-\tfrac{1}{4}\zeta. \tag{14·26}$$

The terms which are neglected in (14·24), (14·25) and (14·26) are $O(1/|\zeta|)$ for large $|\zeta|$. The conditions at the boundary of the hole may be obtained by equating to zero the right-hand side of (12·11). Making use of (14·21) we thus have on $\zeta\bar{\zeta} = a^2$

$$\Delta(\zeta)+\zeta\bar{\Delta}'(\bar{\zeta})+\bar{\delta}'(\bar{\zeta})-\frac{a^4}{\zeta^3}-\frac{5\zeta}{4}+\frac{\zeta^3}{a^2}-\frac{2\zeta^5}{a^4} = 0. \tag{14·27}$$

Moreover, since the integral terms in (10·25) (with $(\zeta, \bar{\zeta})$ written for $(z, \bar{z})$), are single-valued, and since the resultant force at the hole is zero, the conditions for single-valued stresses and displacements reduce to the form (10·22). Proceeding as in the previous example, we find that the complex potential functions satisfying (14·26) and (14·27) and the conditions for single-valued stresses and displacements are given by

$$\left.\begin{aligned} \Delta(\zeta) &= -\frac{1}{8}\left(\zeta+\frac{4a^2}{\zeta}-\frac{8a^4}{\zeta^3}\right), \\ \delta(\zeta) &= \frac{1}{4}\left(\zeta^2+6a^2\log\zeta+\frac{3a^4}{\zeta^2}-\frac{5a^6}{\zeta^4}\right). \end{aligned}\right\} \tag{14·28}$$

These relations, with (12·11) and (12·12), are sufficient to determine ${}^1\phi$ and ${}^1D'$ since the first approximation functions $\Omega(\zeta)$, $\omega(\zeta)$, ${}^0\phi$ and ${}^0D'$ are given by (14·21) and (12·8). The complex stress components referred to co-ordinates in the undeformed body may then be found from equations (11·10). For some purposes, however, it is more useful to know the stress components referred to co-ordinates in the deformed body, and these may be obtained by introducing the expressions obtained for $\partial\,{}^0\phi/\partial\bar{z}$, $\partial\,{}^1\phi/\partial\bar{z}$, ${}^0D$ and ${}^0\bar{D}$ into (12·14). Then remembering (12·5), this process yields

$$\left.\begin{aligned} T^{11} &= T\left\{\left[1-a^2\left(\frac{1}{\bar{\zeta}^2}+2\frac{\zeta}{\bar{\zeta}^3}\right)+3\frac{a^4}{\bar{\zeta}^4}\right]\right. \\ &\qquad \left.-\frac{3Ta^2}{4E}\left[\left(\frac{1}{\bar{\zeta}^2}-\frac{3\zeta}{\bar{\zeta}^3}-\frac{3\zeta^2}{\bar{\zeta}^4}\right)-a^2\left(\frac{5}{\zeta\bar{\zeta}^3}+\frac{2}{\zeta^2\bar{\zeta}^2}-\frac{20\zeta}{\bar{\zeta}^5}\right)-a^4\left(\frac{20}{\bar{\zeta}^6}-\frac{6}{\zeta\bar{\zeta}^5}-\frac{12}{\zeta^2\bar{\zeta}^4}-\frac{1}{\zeta^3\bar{\zeta}^3}\right)-\frac{6a^6}{\zeta^3\bar{\zeta}^5}\right]\right\}, \\ T^{12} &= T\left\{\left[1-a^2\left(\frac{1}{\zeta^2}+\frac{1}{\bar{\zeta}^2}\right)\right]\right. \\ &\qquad \left.+\frac{3Ta^2}{8E}\left[\left(\frac{1}{\zeta^2}+\frac{1}{\bar{\zeta}^2}\right)+a^2\left(\frac{9}{\zeta^2\bar{\zeta}^2}+\frac{4}{\zeta\bar{\zeta}^3}+\frac{4}{\zeta^3\bar{\zeta}}-\frac{2}{\zeta^4}-\frac{2}{\bar{\zeta}^4}\right)-a^4\left(\frac{16}{\zeta^3\bar{\zeta}^3}+\frac{3}{\zeta^2\bar{\zeta}^4}+\frac{3}{\zeta^4\bar{\zeta}^2}\right)+\frac{9a^6}{\zeta^4\bar{\zeta}^4}\right]\right\}, \end{aligned}\right\} \tag{14·29}$$

and from these relations we may obtain by means of (13·5) the physical components of stress referred to the y_α-axes. Moreover, since from (13·4) T^{12} is the sum of the principal (physical) components of stress, it is invariant under rotations of axes, and therefore on the boundary of the hole $\zeta\bar{\zeta} = a^2$ on which the normal and shear stresses are zero, it gives the hoop stress. Introducing polar co-ordinates (r, θ) in the unstrained body by means of the relation $\zeta = r\,e^{i\theta}$ we obtain for the hoop stress

$$[T^{12}]_{r=a} = T\left\{(1-2\cos 2\theta)+\frac{3T}{4E}(1+2\cos 2\theta-2\cos 4\theta)\right\}. \tag{14·30}$$

On the axes of symmetry this expression simplifies to

$$\left.\begin{aligned} [T^{12}]_{r=a} &= -T\left\{1-\frac{3T}{4E}\right\} \quad \text{at} \quad \theta = 0 \quad \text{and} \quad \theta = \pi, \\ [T^{12}]_{r=a} &= 3T\left\{1-\frac{3T}{4E}\right\} \quad \text{at} \quad \theta = \tfrac{1}{2}\pi \quad \text{and} \quad \theta = \tfrac{3}{2}\pi. \end{aligned}\right\} \tag{14·31}$$

It is evident from (14·21) and the first of equations (12·8) that the first approximation displacement function ${}^0D'$ is obtained by replacing $(z, \bar{z})$ by $(\zeta, \bar{\zeta})$ in (14·18). For the second order displacement function ${}^1D'$ we have, from (12·12), (14·21) and (14·28),

$$\begin{aligned}{}^1D'(\zeta, \bar{\zeta}) = \frac{1}{2}\Big\{\zeta - a^2\Big(\frac{1}{\zeta} - \frac{2}{\bar{\zeta}} + \frac{3\zeta}{\bar{\zeta}^2} + \frac{2\zeta^2}{\bar{\zeta}^3}\Big) \\ + \tfrac{1}{3}a^4\Big(\frac{1}{\zeta^3} - \frac{6}{\zeta^2\bar{\zeta}} - \frac{9}{\zeta\bar{\zeta}^2} - \frac{2}{\bar{\zeta}^3} + \frac{27\zeta}{\bar{\zeta}^4}\Big) + \tfrac{1}{5}a^6\Big(\frac{15}{\zeta\bar{\zeta}^4} + \frac{30}{\zeta^2\bar{\zeta}^3} + \frac{5}{\zeta^3\bar{\zeta}^2} - \frac{34}{\bar{\zeta}^5}\Big) - \frac{3a^8}{\zeta^3\bar{\zeta}^4}\Big\}.\end{aligned} \quad (14·32)$$

Employing these expressions for ${}^0D'$ and ${}^1D'$ in (12·3), the real components of displacement in the co-ordinate system x_α may be obtained by the separation of real and imaginary parts. Alternatively, by means of the last of equations (13·10) we may express the results in terms of the polar co-ordinate system (r, θ). At the boundary of the hole $\zeta\bar{\zeta} = a^2$ the displacement components u_r, u_θ in the radial and tangential directions respectively reduce to

$$\left.\begin{aligned}[u_r]_{r=a} &= \frac{3Ta}{4E}\Big\{(1 + 2\cos 2\theta) + \frac{T}{40E}(45 - 40\cos 2\theta + 8\cos 4\theta)\Big\}, \\ [u_\theta]_{r=a} &= -\frac{3Ta}{2E}\Big\{1 - \frac{T}{20E}(5 - \cos 2\theta)\Big\}\sin 2\theta.\end{aligned}\right\} \quad (14·33)$$

15. Extension of an infinite elastic body containing a circular rigid inclusion

We shall now consider the two-dimensional problem in which an elastic body, which is subjected to a uniform tension T in the direction of the y_1- (or x_1-) axis at infinity contains a circular rigid inclusion of radius a. We shall suppose the elastic material to adhere to the inclusion so that at this boundary there is no relative movement of the elastic and rigid bodies. Since the displacement components are zero at the surface of the inclusion, we may choose complex co-ordinates either in the undeformed or in the deformed body. We shall, however, make the latter choice for the co-ordinate system $(z, \bar{z})$ and employ the notation and methods of §§ 9 and 10, since the resulting equations then assume simpler forms.

At the boundary of the inclusion $z\bar{z} = a^2$, the displacement components are zero and we therefore have

$$ {}^nD = {}^n\bar{D} = 0 \quad (n = 0, 1, 2, \ldots), \quad (15·1)$$

while to ensure a uniform tension of magnitude T in the direction of the y_1-axis at infinity we have, as in the previous section, with $\epsilon = T/(4\,{}^0H) = 3T/(4E)$

$$\left.\begin{aligned}\frac{\partial^2({}^0\phi)}{\partial z^2} = \frac{\partial^2({}^0\phi)}{\partial \bar{z}^2} = -\frac{\partial^2({}^0\phi)}{\partial z\,\partial \bar{z}} = -1 + O\Big(\frac{1}{|z|^2}\Big), \\ \frac{\partial^2({}^n\phi)}{\partial z^2} = \frac{\partial^2({}^n\phi)}{\partial \bar{z}^2} = \frac{\partial^2({}^n\phi)}{\partial z\,\partial \bar{z}} = O\Big(\frac{1}{|z|^2}\Big) \quad (n > 0),\end{aligned}\right\} \quad (15·2)$$

for large $|z|$. Expressing ${}^0D$ and ${}^0\phi$ in terms of the complex potential functions $\Omega(z)$ and $\omega(z)$ by means of (10·12), we may write, from (15·1) and (15·2),

$$\Omega(z) - z\bar{\Omega}'(\bar{z}) - \bar{\omega}'(\bar{z}) = 0 \quad \text{on} \quad z\bar{z} = a^2, \quad (15·3)$$

and

$$\bar{\Omega}(\bar{z}) + \bar{z}\Omega'(z) + \omega'(z) = \bar{z} - z + O\Big(\frac{1}{|z|}\Big), \quad (15·4)$$

for large $|z|$. Since there is no resultant force on the inclusion the conditions (10·22) must apply on the circle $z\bar{z} = a^2$ and in order to satisfy (15·4) we may therefore assume expansions of the form (14·8) for $\Omega(z)$ and $\omega(z)$. Introducing these expressions into (15·3), we find that the boundary conditions on $z\bar{z} = a^2$ can only be satisfied if

$$b_r = 0 \quad (r>1), \quad c_r = 0 \quad (r \neq 2),$$
$$A = B = 0,$$
$$b_1 = -a^2, \quad c_2 = \tfrac{1}{2}a^4,$$

so that equations (14·8) reduce to

$$\Omega(z) = \frac{1}{2}\left(z - \frac{2a^2}{z}\right), \quad \omega(z) = -\frac{1}{2}\left(z^2 - \frac{a^4}{z^2}\right). \tag{15·5}$$

Since the boundary conditions at the surface of the inclusion involve the displacement components, the determination of $\Delta(z)$ and $\delta(z)$ is simplified by using the forms (10·16) and (10·18) for ${}^1\phi$ and ${}^1D$. When the solutions (15·5) are substituted in these equations the boundary conditions (15·1) and (15·2) for $n = 1$ become

$$\Delta(z) - z\bar{\Delta}'(\bar{z}) - \bar{\delta}'(\bar{z}) = 0 \quad \text{on} \quad z\bar{z} = a^2, \tag{15·6}$$

and

$$\Delta(z) + z\bar{\Delta}'(\bar{z}) + \bar{\delta}'(\bar{z}) + \tfrac{1}{2}z = O\left(\frac{1}{|z|}\right) \tag{15·7}$$

for large $|z|$. Equations (15·6) and (15·7), together with the conditions (10·22), yield

$$\Delta(z) = -\tfrac{1}{4}z, \quad \delta(z) = 0. \tag{15·8}$$

The stress functions ${}^0\phi$ and ${}^1\phi$, and the stress components $T^{\alpha\beta}$ referred to complex co-ordinates in the deformed body may be found as before from (10·12), (10·16) and (9·13), and the stresses at the surface of the inclusion may then be obtained in a convenient form by means of (13·11). This process yields

$$\left.\begin{aligned} t_{rr} &= \tfrac{1}{2}T\left\{1 + 2\cos 2\theta - \frac{3T}{8E}(3 - 2\cos 4\theta)\right\}, \\ t_{\theta\theta} &= \tfrac{1}{2}T\left\{1 + 2\cos 2\theta + \frac{3T}{8E}(5 - 6\cos 4\theta)\right\}, \\ t_{r\theta} &= -T\sin 2\theta, \end{aligned}\right\} \tag{15·9}$$

where t_{rr}, $t_{\theta\theta}$ and $t_{r\theta}$ are the stress components referred to polar co-ordinates (r, θ) in the deformed body, and evaluated at the surface of the inclusion where $r = a$.

The displacement functions ${}^0D$ and ${}^1D$ are readily obtained from (15·5), (15·8), (10·12) and (10·18), and from these results, with (10·3) and the last of equations (13·11), we may determine the displacement components u_r, u_θ referred to the polar co-ordinate system (r, θ). We thus have

$$\left.\begin{aligned} u_r &= \frac{3Tr}{4E}\left(1 - \frac{a^2}{r^2}\right)^2\left\{\cos 2\theta - \frac{3T}{8E}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\cos 4\theta\right)\right\}, \\ u_\theta &= -\frac{3Tr}{4E}\left(1 - \frac{a^2}{r^2}\right)\left\{1 + \frac{a^2}{r^2} + \frac{3Ta^2}{2Er^2}\left(1 - \frac{a^2}{r^2}\right)\cos 2\theta\right\}\sin 2\theta. \end{aligned}\right\} \tag{15·10}$$

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